Flavor thresholds and Λ in the modified minimal-subtraction scheme

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A discussion of the modified minimal-subtraction ($\overline{\text{MS}}$) prescription in QCD is presented. Effects of flavor thresholds on $\alpha_s(\mu)$ and $\Lambda_{\overline{\text{MS}}}$ are examined. Two parametrization options are described. In the first approach, distinct mass scales $\Lambda_{\overline{\text{MS}}}^{(N_F)}$ associated with N_F, the effective number of flavors, are introduced. In the second scheme, a standard N_F-independent $\Lambda_{\overline{\text{MS}}}$ is adopted but the functional form of $\alpha(\mu)$ is slightly modified.

In quantum chromodynamics (QCD) the $SU(3)_c$ running coupling $\alpha_s(\mu)$ obeys the renormalization-group equation

$$\mu\frac{\partial}{\partial\mu}\alpha_s(\mu)\equiv\beta(\alpha_s)=b_0\alpha_s^{\,2}+b_1\alpha_s^{\,3}+b_2\alpha_s^{\,4}+\cdots\ . \tag{1}$$

The first two terms in the perturbative expansion of $\beta(\alpha_s)$ are renormalization-scheme independent and given by[1,2]

$$b_0=-\frac{1}{2\pi}\left(11-\frac{2N_F}{3}\right)\ , \tag{2a}$$

$$b_1=-\frac{1}{4\pi^2}\left(51-\frac{19N_F}{3}\right)\ , \tag{2b}$$

where N_F is the number of color-triplet quark flavors.[3] Higher-order terms depend on the renormalization prescription.[4] Within the framework of modified minimal subtraction ($\overline{\text{MS}}$),[4,5] b_2 has been computed by Tarasov, Vladimirov, and Zharkov and found to be[6,7]

$$b_2=-\frac{1}{64\pi^3}\left(2857-\frac{5033N_F}{9}+\frac{325N_F^{\,2}}{27}\right)\ . \tag{2c}$$

The β function in Eq. (1) describes the evolution of the effective coupling $\alpha_s(\mu)$ as a function of the mass scale μ (or distance $1/\mu$). It incorporates the large color-charge-screening vacuum-polarization effects controlled by the renormalization group. Of course, the question arises as to what N_F should be employed for a given μ. The answer is provided by convention. In the standard $\overline{\text{MS}}$ scheme, N_F is specified to be the number of quark flavors with mass $\leq\mu$. New flavor thresholds are treated as step functions in $\beta(\alpha_s)$ at $\mu=$quark masses, so that $\alpha_s(\mu)$ is continuous (up to two loops[7]) but its derivative is discontinuous at those values. Real threshold corrections of the form $m_q^{\,2}/Q^2$, where m_q is a generic quark mass, are to be accounted for separately and *not* incorporated into the running coupling. This prescription is somewhat awkward for μ near a quark mass; but it is very convenient for asymptotic values of μ and analysis of grand unified theories.[8]

The step in N_F by one flavor at $\mu=m_q$ rather than, say, $2m_q$ may seem peculiar. One should, however, recall that within the framework of dimensional regularization, μ is merely a unit of mass introduced to keep the bare coupling dimensionless;[4] it is *not* the momentum transfer. The value of μ in the expansion parameter $\alpha_s(\mu)$ appropriate for a particular process is not specified by the $\overline{\text{MS}}$ prescription; nevertheless one expects that μ should generally be at least a factor of 2 smaller than the largest momentum transfer in the process. Suggestions for choosing the optimal μ have been discussed in the literature;[9] this paper will not address that issue.

Formally integrating Eq. (1) yields

$$\frac{1}{\alpha_s(\mu)}=\frac{1}{\alpha_s(\mu_0)}-b_0\ln\left(\frac{\mu}{\mu_0}\right)-\frac{b_1}{b_0}\ln\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)$$
$$-\frac{1}{b_0^{\,2}}(b_2b_0-b_1^{\,2})[\alpha_s(\mu)-\alpha_s(\mu_0)]+O(\alpha_s^{\,2})\ , \tag{3}$$

with μ_0 a reference mass scale. Given $\alpha_s(\mu_0)$ one can compute $\alpha_s(\mu)$ for arbitrary μ by iteratively solving Eq. (3). Of course, N_F must be changed by one unit at each $\mu=m_q$ in the manner described above. So, for example, given an $\alpha_s(\mu_0)$ with $m_c\leq\mu_0\leq m_b$, all other $\alpha_s(\mu)$ corresponding to that domain are determined by iteratively solving Eq. (3) with $N_F=4$. After $\alpha_s(m_c)$ and $\alpha_s(m_b)$ are obtained, $\alpha_s(\mu)$ for $m_s\leq\mu\leq m_c$ and $m_b\leq\mu\leq m_t$ are similarly obtained using $N_F=3$ and 5, respectively.[7] This procedure can be repeated for $\mu\geq m_t$ with $N_F=6$, once m_t is known. For $\mu<m_s$, one might try to continue this evolution using $N_F=2$; however, at those low μ values (effective $m_s\simeq 250$ MeV),[10] the coupling $\alpha_s(\mu)$ is of $O(1)$ and the perturbative expansion in Eqs. (1) and (3) becomes invalid. Results of this iterative procedure are illustrated in Fig. 1.

The above prescription is well defined but cumbersome. It is often more convenient to introduce a reference mass scale Λ which incorporates the iteration procedure and provides a functional form for $\alpha_s(\mu)$. The conventional way of defining $\Lambda_{\overline{\text{MS}}}$ for that purpose and ambiguities due to changing N_F domains will now be discussed.

A μ-independent mass parameter Λ should have the form

$$\Lambda=\mu\exp\left[\frac{1}{b_0\alpha_s(\mu)}-\frac{b_1}{b_0^{\,2}}\ln\left(\frac{-2}{b_0\alpha_s(\mu)}\right)+C\right.$$
$$\left.+\frac{1}{b_0^{\,3}}(b_2b_0-b_1^{\,2})\alpha_s(\mu)+O(\alpha_s^{\,2})\right]\ , \tag{4}$$

where C is an arbitrary constant (which may be viewed as a constant of integration[5]). Using Eq. (1), it is easy to verify that for a given N_F region $\mu(d/d\mu)\Lambda=0$. However, because the b's change as the effective N_F changes, either Λ or C must also exhibit an N_F dependence. One may choose an N_F-independent C (say, $C=0$ for all N_F) and then have

distinct $\Lambda^{(N_F)}$ corresponding to different N_F regions or keep Λ fixed for all N_F by introducing distinct $C^{(N_F)}$ in Eq. (4). Both possibilities will be subsequently examined. In either case, iteratively solving Eq. (4) for $\alpha_s(\mu)$ gives

$$\alpha_s^{-1}(\mu) = -\frac{b_0}{2}\ln(\mu^2/\Lambda^2) + \frac{b_1}{b_0}[\ln\ln(\mu^2/\Lambda^2) - b_0{}^2C/b_1] - \frac{2b_1{}^2}{b_0{}^3}\frac{[\ln\ln(\mu^2/\Lambda^2) - b_0{}^2C/b_1 - (b_2b_0/b_1{}^2-1)]}{\ln(\mu^2/\Lambda^2)} + O\left(\frac{1}{\ln^2(\mu^2/\Lambda^2)}\right), \tag{5}$$

or inverting

$$\alpha_s(\mu) = \frac{-2}{b_0\ln(\mu^2/\Lambda^2)}\left\{1 + \frac{2b_1}{b_0{}^2}\frac{[\ln\ln(\mu^2/\Lambda^2) - b_0{}^2C/b_1]}{\ln(\mu^2/\Lambda^2)} + \frac{4b_1{}^2}{b_0{}^4\ln^2(\mu^2/\Lambda^2)}\left[\left\{\ln\ln(\mu^2/\Lambda^2) - \frac{b_0{}^2C}{b_1} - \frac{1}{2}\right\}^2 + \frac{b_2b_0}{b_1{}^2} - \frac{5}{4}\right] + O\left(\frac{1}{\ln^3(\mu^2/\Lambda^2)}\right)\right\}. \tag{6}$$

For all applications known to the author, only the first two terms in these expressions are retained. That is generally a good approximation as long as μ/Λ is large. The extra terms have been retained in Eqs. (5) and (6) to provide a measure of the truncation error and the validity of the expansion.

The conventional definition of $\Lambda_{\overline{\rm MS}}$ (given by Bardeen, Buras, Duke, and Muta[5]) sets $C=0$ in Eqs. (4), (5), and (6). This simplifies somewhat the functional form of $\alpha_s(\mu)$. Unfortunately, as mentioned before, to do so for all N_F requires distinct $\Lambda_{\overline{\rm MS}}^{(N_F)}$ for different effective flavor regimes. Putting numbers into Eq. (4) with $C=0$ gives[7]

$$\Lambda_{\overline{\rm MS}}^{(6)} = \mu\exp\left\{-\frac{2}{7}\left[\frac{\pi}{\alpha_s(\mu)} - \frac{13}{14}\ln\left(\frac{4\pi}{7\alpha_s(\mu)}\right) - \frac{8165}{4704}\frac{\alpha_s(\mu)}{\pi}\right]\right\}, \quad \mu > m_t, \tag{7a}$$

$$\Lambda_{\overline{\rm MS}}^{(5)} = \mu\exp\left\{\frac{-6}{23}\left[\frac{\pi}{\alpha_s(\mu)} - \frac{29}{23}\ln\left(\frac{12\pi}{23\alpha_s(\mu)}\right) - \frac{91581}{152352}\frac{\alpha_s(\mu)}{\pi}\right]\right\}, \quad m_b < \mu \leq m_t, \tag{7b}$$

$$\Lambda_{\overline{\rm MS}}^{(4)} = \mu\exp\left\{\frac{-6}{25}\left[\frac{\pi}{\alpha_s(\mu)} - \frac{77}{50}\ln\left(\frac{12\pi}{25\alpha_s(\mu)}\right) + \frac{51687}{180000}\frac{\alpha_s(\mu)}{\pi}\right]\right\}, \quad m_c < \mu \leq m_b, \tag{7c}$$

$$\Lambda_{\overline{\rm MS}}^{(3)} = \mu\exp\left\{-\frac{2}{9}\left[\frac{\pi}{\alpha_s(\mu)} - \frac{16}{9}\ln\left(\frac{12\pi}{27\alpha_s(\mu)}\right) + \frac{2641}{2592}\frac{\alpha_s(\mu)}{\pi}\right]\right\}, \quad m_s < \mu \leq m_c. \tag{7d}$$

Relationships between $\Lambda_{\overline{\rm MS}}^{(N_F)}$ pairs are obtained using the continuity of $\alpha_s(\mu)$ at quark masses. For example, given a value for $\alpha_s(m_b)$, one determines both $\Lambda_{\overline{\rm MS}}^{(4)}$ and $\Lambda_{\overline{\rm MS}}^{(5)}$ from Eqs. (7c) and (7b). It is also convenient to have some explicit approximate relationships between these mass scales. Again using continuity and iteration, one finds from Eq. (5)

$$\Lambda_{\overline{\rm MS}}^{(6)} \simeq \Lambda_{\overline{\rm MS}}^{(5)}(\Lambda_{\overline{\rm MS}}^{(5)}/m_t)^{2/21}[\ln(m_t{}^2/\Lambda_{\overline{\rm MS}}^{(5)\,2})]^{-321/3381}, \tag{8a}$$

$$\Lambda_{\overline{\rm MS}}^{(5)} \simeq \Lambda_{\overline{\rm MS}}^{(6)}(m_t/\Lambda_{\overline{\rm MS}}^{(6)})^{2/23}[\ln(m_t{}^2/\Lambda_{\overline{\rm MS}}^{(6)\,2})]^{321/3703}, \tag{8b}$$

$$\Lambda_{\overline{\rm MS}}^{(5)} \simeq \Lambda_{\overline{\rm MS}}^{(4)}(\Lambda_{\overline{\rm MS}}^{(4)}/m_b)^{2/23}[\ln(m_b{}^2/\Lambda_{\overline{\rm MS}}^{(4)\,2})]^{-963/13225}, \tag{8c}$$

$$\Lambda_{\overline{\rm MS}}^{(4)} \simeq \Lambda_{\overline{\rm MS}}^{(5)}(m_b/\Lambda_{\overline{\rm MS}}^{(5)})^{2/25}[\ln(m_b{}^2/\Lambda_{\overline{\rm MS}}^{(5)\,2})]^{963/14375}, \tag{8d}$$

$$\Lambda_{\overline{\rm MS}}^{(4)} \simeq \Lambda_{\overline{\rm MS}}^{(3)}(\Lambda_{\overline{\rm MS}}^{(3)}/m_c)^{2/25}[\ln(m_c{}^2/\Lambda_{\overline{\rm MS}}^{(3)\,2})]^{-107/1875}, \tag{8e}$$

$$\Lambda_{\overline{\rm MS}}^{(3)} \simeq \Lambda_{\overline{\rm MS}}^{(4)}(m_c/\Lambda_{\overline{\rm MS}}^{(4)})^{2/27}[\ln(m_c{}^2/\Lambda_{\overline{\rm MS}}^{(4)\,2})]^{107/2025}. \tag{8f}$$

These mass scales will turn out to be quite different in magnitude. To illustrate this point, take $\Lambda_{\overline{\rm MS}}^{(4)} = 100$ MeV, the central value obtained from the radiative Υ decay branching ratio,[11] and assume $m_t \simeq 35$ GeV. For those values, one finds (approximately)

$$\Lambda_{\overline{\rm MS}}^{(6)}:\Lambda_{\overline{\rm MS}}^{(5)}:\Lambda_{\overline{\rm MS}}^{(4)}:\Lambda_{\overline{\rm MS}}^{(3)} :: 27:63:100:130. \tag{9}$$

Rather than requiring $C=0$ for all N_F and thereby rendering $\Lambda_{\overline{\rm MS}}^{(N_F)}$ dependent on N_F, an alternate strategy is to employ a single "standard" $\Lambda_{\overline{\rm MS}}$ and retain N_F-dependent

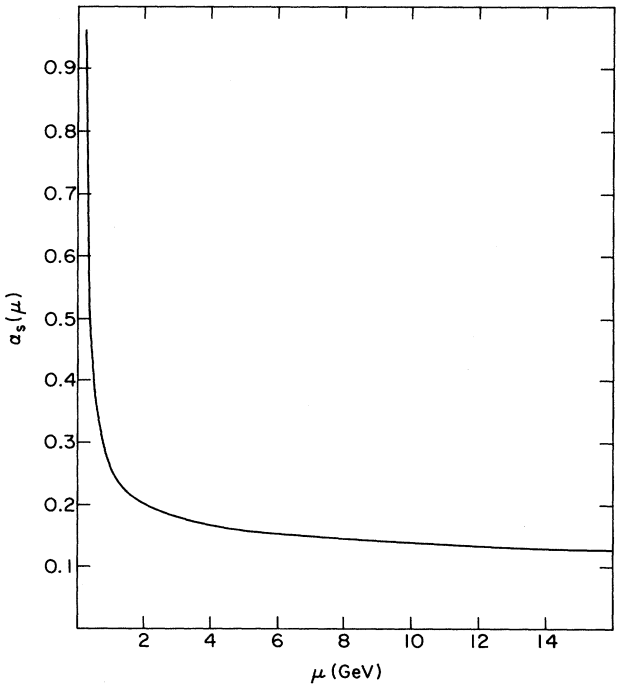

FIG. 1. $\alpha_s(\mu)$ as a function of μ using $\alpha_s(2\text{ GeV}) = 0.20$ as input.

$C^{(N_F)}$ in Eqs. (4), (5), and (6). Having a standard $\Lambda_{\overline{\rm MS}}$ may help to avoid confusion and provide a useful means of comparing different experiments. In addition, a glance at Eqs. (5) and (6) indicates that $C^{(N_F)}$ does not significantly complicate the form of $\alpha_s(\mu)$. Indeed, the benefit of having an N_F-independent $\Lambda_{\overline{\rm MS}}$ in the logarithmic arguments of those expressions would seem to more than compensate. Specifying a standard $\Lambda_{\overline{\rm MS}}$, the $C^{(N_F)}$ are [from Eq. (4)] given by

$$C^{(N_F)} = \ln(\Lambda_{\overline{\rm MS}}/\Lambda_{\overline{\rm MS}}^{(N_F)}) \ . \tag{10}$$

If one of the $\Lambda_{\overline{\rm MS}}^{(N_F)}$ is chosen to be the standard, then its corresponding $C^{(N_F)}$ is zero, while the others are simply determined from Eq. (10) and the relationships in Eq. (8). [For a more precise determination Eq. (5), including its higher-order terms, can be employed.] One might be inclined at this time to choose $\Lambda_{\overline{\rm MS}}^{(4)} = \Lambda_{\overline{\rm MS}}$ as the standard, since it is currently the most accurately measured one.[11] In that case, the $N_F = 3, 4, 5$ parametrizations are given up to terms of $O(1/\ln^3(\mu^2/\Lambda_{\overline{\rm MS}}{}^2))$ by

$$\alpha_s(\mu) = \frac{12\pi}{27\ln(\mu^2/\Lambda_{\overline{\rm MS}}{}^2)}\left[1 - \frac{\frac{64}{81}\ln\ln(\mu^2/\Lambda_{\overline{\rm MS}}{}^2) + 2C^{(3)}}{\ln(\mu^2/\Lambda_{\overline{\rm MS}}{}^2)}\right], \quad m_s < \mu \leq m_c \ , \tag{11a}$$

$$\alpha_s(\mu) = \frac{12\pi}{25\ln(\mu^2/\Lambda_{\overline{\rm MS}}{}^2)}\left[1 - \frac{\frac{462}{625}\ln\ln(\mu^2/\Lambda_{\overline{\rm MS}}{}^2) + 2C^{(4)}}{\ln(\mu^2/\Lambda_{\overline{\rm MS}}{}^2)}\right], \quad m_c < \mu \leq m_b \ , \tag{11b}$$

$$\alpha_s(\mu) = \frac{12\pi}{23\ln(\mu^2/\Lambda_{\overline{\rm MS}}{}^2)}\left[1 - \frac{\frac{348}{529}\ln\ln(\mu^2/\Lambda_{\overline{\rm MS}}{}^2) + 2C^{(5)}}{\ln(\mu^2/\Lambda_{\overline{\rm MS}}{}^2)}\right], \quad m_b < \mu \leq m_t \ , \tag{11c}$$

where for $\Lambda_{\overline{\rm MS}} = \Lambda_{\overline{\rm MS}}^{(4)}$

$$C^{(3)} = \ln(\Lambda_{\overline{\rm MS}}/\Lambda_{\overline{\rm MS}}^{(3)}) \simeq -\tfrac{1}{27}[\ln(m_c{}^2/\Lambda_{\overline{\rm MS}}{}^2) + \tfrac{107}{75}\ln\ln(m_c{}^2/\Lambda_{\overline{\rm MS}}{}^2)] \ , \tag{12a}$$

$$C^{(4)} = \ln(\Lambda_{\overline{\rm MS}}/\Lambda_{\overline{\rm MS}}^{(4)}) = 0 \ , \tag{12b}$$

$$C^{(5)} = \ln(\Lambda_{\overline{\rm MS}}/\Lambda_{\overline{\rm MS}}^{(5)}) \simeq \tfrac{1}{23}[\ln(m_b{}^2/\Lambda_{\overline{\rm MS}}{}^2) + \tfrac{963}{575}\ln\ln(m_b{}^2/\Lambda_{\overline{\rm MS}}{}^2)] \ . \tag{12c}$$

In the future $\Lambda_{\overline{\rm MS}}^{(6)}$ will probably become a more natural standard (after m_t is known). It is smaller than the others [see Eq. (9)]; a feature which should reduce somewhat the truncation errors in $\alpha_s(\mu)$ when only terms up to $O(1/\ln^2(\mu^2/\Lambda_{\overline{\rm MS}}{}^2))$ are retained as in Eq. (11).

A more radical approach would be to choose a convenient scale, say, $\Lambda_{\overline{\rm MS}} = 50$ MeV, as the standard and keep all $C^{(N_F)} \neq 0$. Although that would certainly cause confusion, it might have some benefits. For example, one might invoke a value that minimizes the truncation error in $\alpha_s(\mu)$ or simplifies the $C^{(N_F)}$.

Which of the above parametrizations will prove more popular is not clear. The first scheme has a simpler $\alpha_s(\mu)$ form, since $C^{(N_F)} = 0$; however, its distinct $\Lambda_{\overline{\rm MS}}^{(N_F)}$ may be cumbersome. The second approach has the advantage of requiring a single "standard" $\Lambda_{\overline{\rm MS}}$ at the expense of introducing $C^{(N_F)}$ terms in the $\alpha_s(\mu)$ expansion. Time will tell which of these parametrization options will become the more accepted convention.

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$$\alpha_s(\mu) \to \alpha_s(\mu)\{1 + 7N_F[\alpha_s(\mu)/\pi]^2/72\} \ .$$

In this latter approach, b_2 in Eq. (2c) is replaced by

$$-\frac{1}{64\pi^3}(2857 - 525N_F + \tfrac{269}{27}N_F{}^2) \ .$$

The effect of this shift is incorporated in Eq. (7). Numerically, it is insignificant.

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