QUANTENCHROMODYNAMIK

Vorlesungsprogramm:

1.) Grundlagen der QCD: Farbfreiheitsgrade der Quarks
nicht-abel'sche Feldtheorie der Quarks und Gluonen
asymptotische Freiheit

2.) QCD bei kurzen Abständen: Nukleon-Strukturfunktionen
e^+e^- Annihilation in Hadronen
Drell-Yan Prozesse
Jet-Physik in e^+e^- Annihilation und Hadron-Hadron-Stößen
Quarkonium-Physik

3.) QCD bei großen Abständen: Gittereichfeldtheorie der QCD
QCD-Vakuum

http://people.web.psi.ch/spira/vorlesung/qcd/

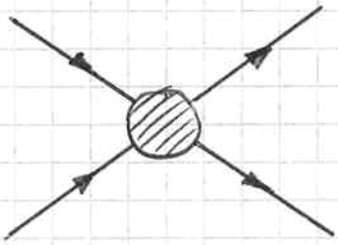
A. GRUNDLAGEN DER QCD

§1. Einführung von Color

QCD: feldtheoretische Formulierung der starken Ww.

historische Definition der starken Ww:

- Bindungskraft der Nukleonen im Kern
- Kraft bei Nukleon-Nukleon-Stößen



Wwsdistanz: $d \sim 1\,\mathrm{fm} \rightarrow \sigma \sim 4\pi d^2 \sim 10\,\mathrm{mb}$

Wwsstärke: $V(R) = \frac{g_s^2}{4\pi} e^{-\frac{R}{d}}$

$$\frac{g_s^2}{4\pi} \sim 10^2 \frac{g_{em}^2}{4\pi} \sim 1$$

Spin-Statistik Problem des Quarkmodells

$\Delta^{++}(s_z = \frac{3}{2}) = u(\Uparrow)\,u(\Uparrow)\,u(\Uparrow)$ $\leftarrow$ total symm. Spinwellenfkt.

$\in$ Dekuplett $\uparrow$ Fermistatistik: total antisymm. Gesamtwellenfkt.

(i) Grundzustand $\neq$ rel. S-Wellen-Kombination ↯ zur naiven Erfahrung

P-Wellen $\rightarrow$ Knoten $\rightarrow$ verbotene Zonen

$\rightarrow$ erhöhte Energie infolge Unschärfe

(ii) Magnetische Momente von Nukleonen

$$\vec{\mu} = \frac{eQ}{2m}\left[\vec{l} + 2\vec{S}\right]$$

S-Wellen $l=0$: Nukleonmomente setzen sich additiv aus Quarkmomenten zusammen

$$\mu_N = \langle N | \sum_{i=1}^{3} \mu(i)\, \sigma_3(i) | N \rangle$$

aufgrund der Spin-Wellenfkt: $[\mu_u = -2\mu_d]$

$$\left.\begin{aligned} \mu_p &= \frac{4}{3}\mu_u - \frac{1}{3}\mu_d = -\left(\frac{8}{3}+\frac{1}{3}\right)\mu_d = -3\mu_d \quad \text{für } m_u \approx m_d \\ \mu_n &= \frac{4}{3}\mu_d - \frac{1}{3}\mu_u = \left(\frac{4}{3}+\frac{2}{3}\right)\mu_d = 2\mu_d \end{aligned}\right\}$$

Clebsch-Gordan

Verhältnis: $\boxed{\frac{\mu_p}{\mu_n} = -\frac{3}{2} \qquad \exp = -1.46}$ kein $\ell \neq 0$ Strom-Beitrag erforderlich

effektive Quarkmasse: $\mu_p = \frac{e}{2m_p} 2.79 = -\frac{1}{3}\frac{e}{2m_d}(-3) = \frac{e}{2m_d}$

$$\Longrightarrow \boxed{m_q^{eff} = \frac{m_p}{2.79} \approx 330\ \text{MeV}}$$

<u>Lösung</u>: Quarks tragen 3wertiges Unterscheidungsmerkmal, so daß symm. Quarkmodell möglich

I. Color-Hypothese (Greenberg '64)

Neben Flavorladungen tragen Quarks Farbladungen; jedes Quark kommt in genau 3 Farben vor (rot, blau, grün = 1,2,3):

$q = (q_1, q_2, q_3)$

<u>Color-Transformationen</u>: maximale Mischungsgruppe der drei Farbfreiheitsgrade [≠ gemeinsame Phase]

$$q \to q' = e^{i\sum_{k=1}^{8} \alpha_k \frac{\lambda_k}{2}}\, q \quad \Leftarrow SU(3)_c \text{ Transformationen}$$

= unimodulare, unitäre 3x3 Matrizen

[nicht-abel'sche Gruppe]

Gell-Mann Matrizen: $\lambda_k \quad k=1,2,\dots,8$ [3dim Erweiterung der $\vec{\sigma}$ in SU_2]

$$\lambda_k^+ = \lambda_k \Longrightarrow e^{-i\alpha_k \frac{\lambda_k}{2}} \text{ unitär: } U^+U = \mathbb{1}$$

$$\mathrm{Tr}\,\lambda_k = 0 \Longrightarrow \text{unimodular: } \det U = +1$$

explizite Darstellung:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Eigenschaften:

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = i f_{ijk} \frac{\lambda_k}{2} \quad [A_2 \text{ Algebra}]$$

$$\left\{\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right\} = \frac{1}{3}\delta_{ij}\mathbb{1} + d_{ijk}\frac{\lambda_k}{2}$$

$$\mathrm{Tr}(\lambda_i \lambda_j) = 2\delta_{ij} \qquad \mathrm{Tr}(\lambda_i) = 0$$

I! Color-Hypothese (Gell-Mann '72)

Die $SU(3)_c$ Symmetrie ist exakt. Alle physikalischen (freien) Zustände, Observablen und Wechselwirkungen sind $SU(3)_c$ Singuletts

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(a) Quarks als Farb-Tripletts treten nicht als freie Teilchen auf

(b) Farb-Wfkt. Baryon: $\frac{1}{\sqrt{6}}\varepsilon_{ijk}$, Meson: $\frac{1}{\sqrt{3}}\delta_{ij}$ $\Big\}$ $\varepsilon_{ijk}, \delta_{ij}$ $SU(3)_c$ Singuletts

The Nonvanishing Values of f_{ijk} and d_{ijk}

(ijk)	f_{ijk}	(ijk)	d_{ijk}
123	1	118	$1/\sqrt{3}$
147	$\frac{1}{2}$	146	$\frac{1}{2}$
156	$-\frac{1}{2}$	157	$\frac{1}{2}$
246	$\frac{1}{2}$	228	$1/\sqrt{3}$
257	$\frac{1}{2}$	247	$-\frac{1}{2}$
345	$\frac{1}{2}$	256	$\frac{1}{2}$
367	$-\frac{1}{2}$	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	$\frac{1}{2}$
678	$\sqrt{3}/2$	355	$\frac{1}{2}$
		366	$-\frac{1}{2}$
		377	$-\frac{1}{2}$
		448	$-1/2\sqrt{3}$
		558	$-1/2\sqrt{3}$
		668	$-1/2\sqrt{3}$
		778	$-1/2\sqrt{3}$
		888	$-1/\sqrt{3}$

Bsp: $\Delta^{++}\left(s_z=+\frac{3}{2}\right)=\frac{1}{\sqrt{6}}\,\varepsilon_{ijk}\,u_i(\uparrow)\,u_j(\uparrow)\,u_k(\uparrow)$

$\Phi\,(s_z=+1)=\frac{1}{\sqrt{3}}\,\delta_{ij}\,s_i(\uparrow)\,\bar{s}_j(\uparrow)$

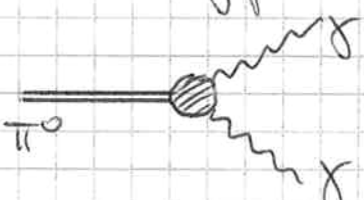
(c) elm. Ww.: $\mathcal{L}_{elm}=-e\,j^\mu A_\mu$

$$j_\mu=\sum_{fl}\bar{q}\,\gamma_\mu Q_q\, q\equiv\sum_{fl}\sum_{c}\bar{q}_c\,\gamma_\mu Q_q\, q_c$$

SU(3) Singulett

TESTS DER COLOR-HYPOTHESE:

1.) $\pi^0\to\gamma\gamma$ Zerfall

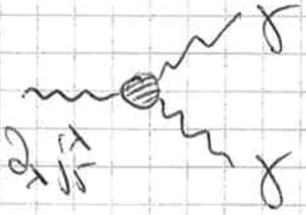


$\mathcal{M}(\pi^0\to\gamma\gamma)=i\,\varepsilon^*_\mu(k_1)\,\varepsilon^*_\nu(k_2)\,T_{\mu\nu}(k_1,k_2;p)$

Lorentzinv.
Paritätsinv. $\Big\}$ $T_{\mu\nu}(k_1,k_2;p)=\varepsilon_{\mu\nu\alpha\beta}\,k_1^\alpha k_2^\beta\, T(p^2=m_\pi^2)$

[π^0 = Pseudoskalar]

Green's Fkt:



$$j_5^\lambda=\bar{q}\,\gamma^\lambda\gamma_5\frac{\lambda^3}{2}\,q$$
$$=\frac{1}{2}\bar{u}\,\gamma^\lambda\gamma_5\, u-\frac{1}{2}\bar{d}\,\gamma^\lambda\gamma_5\, d$$

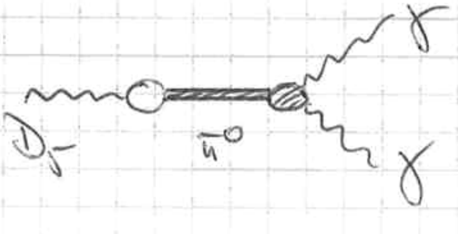
$\partial_\lambda j_5^\lambda\sim\pi^0$ Quantenfeld, farbsummiert

$=\varepsilon_{\mu\nu\alpha\beta}\,k_1^\alpha k_2^\beta\, A(p^2)$: $\quad p_\lambda\langle j_5^\lambda j^\mu j^\nu\rangle\sim\langle\partial_\lambda j_5^\lambda j^\mu j^\nu\rangle$

Eigenschaften von $A(p^2)$:

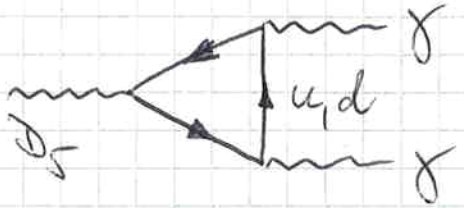
(i) $A(p^2=0)=0$ [aus p_λ * Invariantenzerleg $\langle j_5^\lambda j^\mu j^\nu\rangle$]

(ii) $p^2\to m_\pi^2$:



$$=-\frac{m_\pi^2\frac{f_\pi}{\sqrt{2}}\,T(\pi^0\to\gamma\gamma)}{p^2-m_\pi^2}$$ [PCAC]

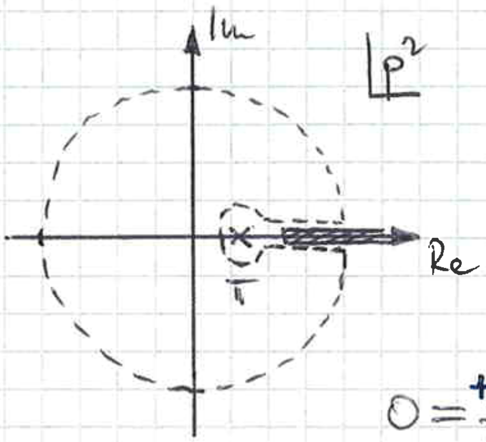
(iii) Mehrteilchen-Zwischenzustände $\sim \frac{\Sigma f}{(3m_\pi)^2}$ ← vernachlässigbar für $p^2 \to 0$ [~10%]

(iv) $p^2 \to \infty$: 

$$= \frac{e^2}{2\pi^2}\,\frac{1}{2}\left(Q_u^2 - Q_d^2\right) N_c$$

$N_c = \#$ Farben

Dispersionsrelation für $A(p^2)$:



Cauchy: $$A(0) = 0 = \oint dp^2\, \frac{A(p^2)}{p^2} = \int_{\text{cut}} \frac{dp^2}{p^2}\, \frac{m_\pi^2 \frac{-f_\pi}{\sqrt{2}} T}{p^2 - m_\pi^2} + 2\pi i A_\infty$$

$$0 = \frac{+f_\pi}{\sqrt{2}}\, T(\pi^0 \to \gamma\gamma) + \frac{e^2}{4\pi^2}\left(Q_u^2 - Q_d^2\right) N_c$$

Breite: $$\boxed{\Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha^2}{32\pi^3}\,\frac{m_\pi^3}{f_\pi^2}\left(Q_u^2 - Q_d^2\right)^2 N_c^2}$$

ohne Farbe $N_c = 1$: $\Gamma = 0.868 \pm 0.065\ \text{eV}$

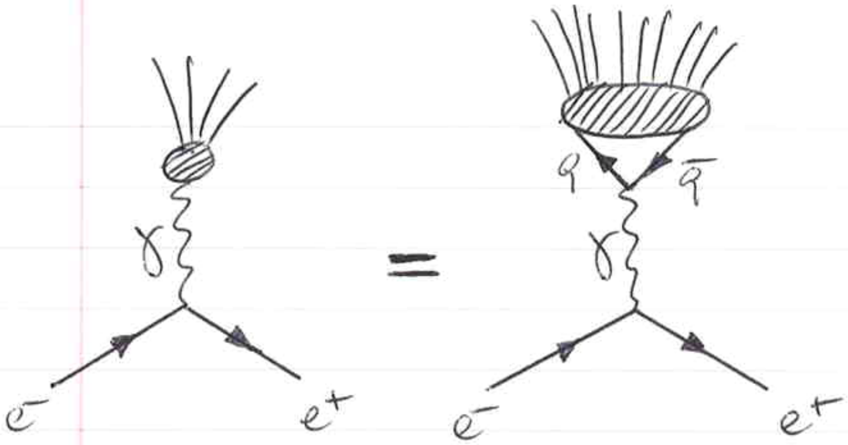
mit Farbe $N_c = 3$: $\Gamma = 7.81 \pm 0.60\ \text{eV}$

experimentelle : $\Gamma_{exp} = 7.84 \pm 0.56\ \text{eV}$

ok

2.) $e^+e^- \to$ Hadronen

Im Quark-Partonmodell wird Produktionswahrscheinlichkeit in $e^+e^- \to$ Hadronen durch diejenige aller $q\bar{q}$ Paare gegeben; Endzustandwechselwirkung vernachlässigbar für $\frac{d_{\text{Prod.}\,q\bar{q}}}{d_{\text{hadron}}} \sim \frac{1\,\text{GeV}}{E} \to 0 \quad (E \to \infty)$



$$\boxed{R = \frac{\sigma(e^+e^- \to \text{Hadr.})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{fl,c} \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{fl} e_q^2}$$

q	e_q
u, c, t	$+\frac{2}{3}$
d, s, b	$-\frac{1}{3}$

Energie	produz. Quarks	R ohne Color	R mit Color
< 3 GeV	u, d, s	$\frac{4}{9}+\frac{1}{9}+\frac{1}{9}=\frac{2}{3}$	2
> 5 GeV	$+c$	$\frac{6}{9}+\frac{4}{9}=\frac{10}{9}$	$\frac{10}{3}$
> 10 GeV	$+b$	$\frac{10}{9}+\frac{1}{9}=\frac{11}{9}$	$\frac{11}{3}$

§2. Gluon-Eichfelder

In Analogie zur QED:

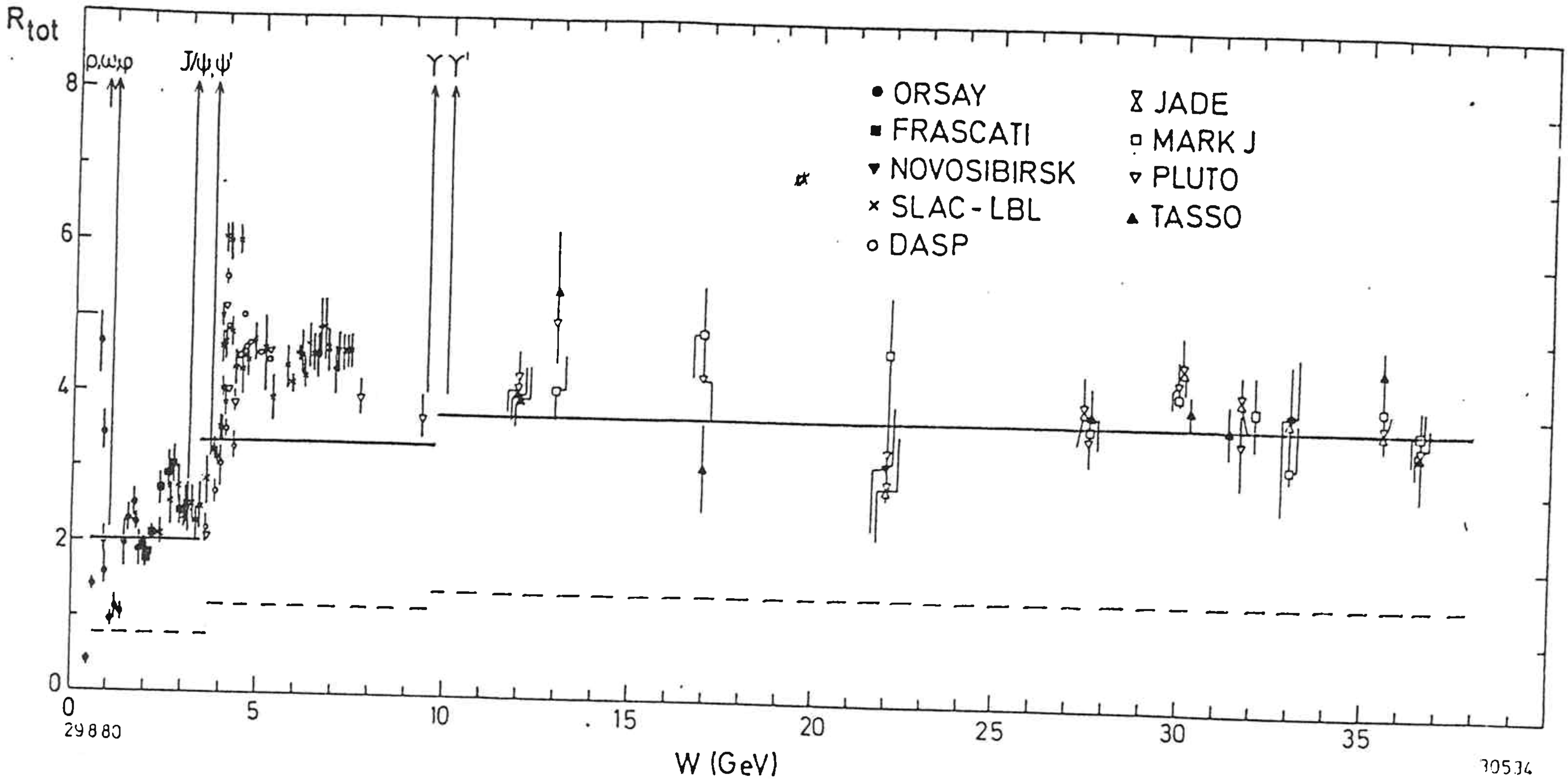
II. Color-Hypothese (Nambu '66, Fritzsch + Gell-Mann '72, Leutwyler '73)

Color-Ladungen sind Quellen von Eichfeldern ($\Rightarrow$ Gluonen), welche die starke Wechselwirkung zwischen Quarks aufbauen.

Lagrangedichte für Farbtriplett:

$$\mathcal{L}_q = \bar{q}(x)\left(i\not{\partial} - m_q\right) q(x) \qquad \text{mit } q = (q_1, q_2, q_3)$$

$$m_{q_1} = m_{q_2} = m_{q_3} \quad SU(3)_c \text{ Singulett}$$



R_tot
ρ,ω,φ
J/ψ,ψ'
Υ Υ'
• ORSAY
▪ FRASCATI
▾ NOVOSIBIRSK
× SLAC - LBL
○ DASP
JADE
□ MARK J
▿ PLUTO
▲ TASSO
0
2
4
6
8
0
5
10
15
20
25
30
35
W (GeV)
29880
30534

9

– invariant unter globalen, nicht-abel'schen $SU(3)_c$–Trf.

$$\left.\begin{aligned} q(x) &\to S\,q(x) \\ \bar{q}(x) &\to \bar{q}(x)\,S^{-1} \end{aligned}\right\} S = e^{-i\alpha_k T^k} \qquad \left(T^k = \frac{\lambda_k}{2}\right)$$

– nicht invariant unter lokalen $SU(3)_c$–Trf.: $\alpha_k = \alpha_k(x)$

$$\mathcal{L}_q \to \mathcal{L}_q + \bar{q}(x)\left(S^{-1} i \not\partial S\right) q(x)$$

lokal eichinvariant gemacht durch Einführung von 8 minimal gekoppelten Gluonfeldern $G^k_\mu(x)$ $(k=1,\dots,8)$
(Gluon-Matrix $G_\mu = G^k_\mu T^k$)

$$i\partial_\mu \to i\partial_\mu - g_s G_\mu = i D_\mu$$

$$\mathcal{L}_q = \bar{q}(x)\left(i\not D - m_q\right) q(x) = \bar{q}(x)\left(i\not\partial - m_q - g_s \not G(x)\right) q(x)$$

mit
$$\begin{aligned} q(x) &\to S(x)\,q(x) \qquad\qquad \alpha_k = \alpha_k(x) \\ \bar{q}(x) &\to \bar{q}(x)\,S^{-1} \\ G_\mu(x) &\to \underbrace{S G_\mu S^{-1}}_{\text{ROTAT.}} - \underbrace{\frac{i}{g_s} S\partial_\mu S^{-1}}_{\text{TRANSL.}} \end{aligned}$$

kovar. Ableitung: $i\not D q \to i\not D' q' = \left[i\not\partial - g_s S \not G S^{-1} - i(\not\partial S)S^{-1}\right] S q$
$[\partial_\mu(SS^{-1}) = 0]$
$$= S\left(i\not\partial - g_s \not G\right) q = S i \not D S^{-1} S q$$

$$\underline{\underline{D \to D' = S D S^{-1}}} \quad \text{ROTATION}$$

Gluonfeld-Lagrangedichte:

Wirbel: $G_{\mu\nu} = D_\nu G_\mu - D_\mu G_\nu = \partial_\nu G_\mu - \partial_\mu G_\nu - i g_s [G_\mu, G_\nu]$

Eich-Trf.: $G_{\mu\nu} \to G'_{\mu\nu} = S G_{\mu\nu} S^{-1}$ reine Rotation [keine Observable]

aus $G_{\mu\nu} = \frac{i}{g_s}[D_\mu, D_\nu]$

$$\mathcal{L}_g = -\frac{1}{2}\mathrm{Tr}\, G_{\mu\nu}^2 = -\frac{1}{4}\left(G_{\mu\nu}^k\right)^2$$ ← eichinvariant: kein Massenterm $\left(+\frac{1}{2} m_g^2 \mathrm{Tr}\, G_\mu^2\right)$

↑

besteht aus: (a) kin. Teil $= -\frac{1}{4}\left(\partial_\nu G_\mu^k - \partial_\mu G_\nu^k\right)^2$

(b) trilineare Kopplung $\sim g_s GGG$

(c) quartische Kopplung $\sim g_s^2 GGGG$

- Selbstwechselwirkung des Gluonfeldes: farbgeladene Gluonen sind Quellen für Gluonen ($\neq \gamma$)

- g_s universell, im Eichsektor fixiert: Farbladungen quantisiert

<u>Lagrangedichte I der QCD:</u>

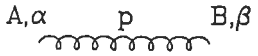
$$\mathcal{L} = \bar{q}\left(i\not{D} - m_q\right)q - \frac{1}{2}\mathrm{Tr}\, G_{\mu\nu}^2$$

$$= \bar{q}\left(i\not{\partial} - m_q\right)q - \frac{1}{2}\mathrm{Tr}\left(\partial_\nu G_\mu - \partial_\mu G_\nu\right)^2$$ kinet. Teil

$$- g_s \bar{q}\not{G} q$$ Quark-Gluon Kopplg.

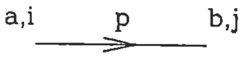
$$+ i g_s \mathrm{Tr}\left(\partial_\nu G_\mu - \partial_\mu G_\nu\right)\left[G_\mu, G_\nu\right]$$ 3Gluon-Kopplg.

$$+ \frac{g_s^2}{2}\mathrm{Tr}\left[G_\mu, G_\nu\right]^2$$ 4Gluon-Kopplg.

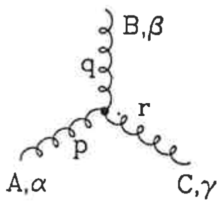


$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^{\alpha}p^{\beta}}{p^{2}+i\epsilon}\right]\frac{i}{p^{2}+i\epsilon}$$

$$\delta^{AB}\frac{i}{(p^{2}+i\epsilon)}$$

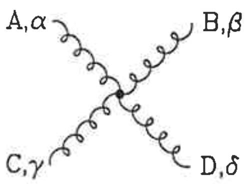


$$\delta^{ab}\frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

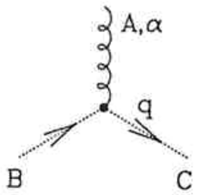


$$-g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

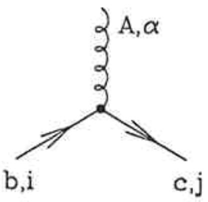
(all momenta incoming, $p+q+r = 0$)



$$-ig^{2}\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^{2}\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^{2}\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$



$$g\ f^{ABC}q^{\alpha}$$



$$-ig\ (t^{A})_{cb}\ (\gamma^{\alpha})_{ji}$$

§3. Feynman-Pfadintegrale

QM: Übergangsamplitude eines Teilchens $\{x_0,t_0\} \to \{x,t\}$:

$$\langle x,t|x_0,t_0\rangle = \langle x|e^{-iH(t-t_0)}|x_0\rangle = \langle x|e^{-iH\varepsilon}e^{-iH\varepsilon}\cdots|x_0\rangle$$

$$= \int \prod_i dx_i \, \langle x|e^{-iH\varepsilon}|x_n\rangle\langle x_n|\cdots|x_0\rangle$$

$$\mathbb{1} = \int dx_i \, |x_i\rangle\langle x_i|$$

$$\langle y_2|e^{-iH\varepsilon}|y_1\rangle = \langle y_2|e^{-i\frac{\vec{p}^2}{2m}\varepsilon}|y_1\rangle e^{-iV(y_1)\varepsilon} \sim e^{i\mathcal{L}\varepsilon}$$

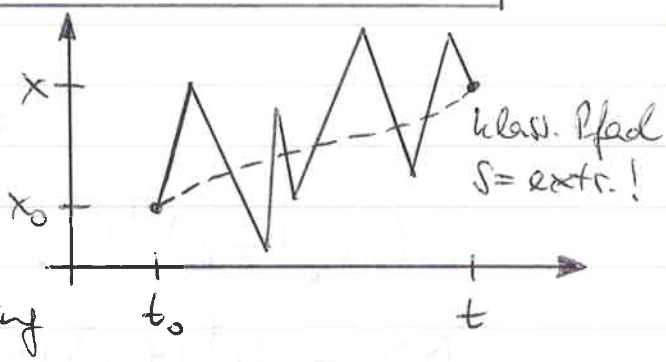
$$\sim \int dk\, e^{-i\frac{k^2}{2m}\varepsilon}\langle y_2|k\rangle\langle k|y_1\rangle \sim \int dk\, e^{-i\frac{k^2}{2m}\varepsilon + i(y_2-y_1)k}$$

$$\sim \int dk \exp\left\{-i\frac{\varepsilon}{2m}\left(k - m\frac{y_2-y_1}{\varepsilon}\right)^2 + i\frac{m}{2\varepsilon}(y_2-y_1)^2\right\}$$

$$\sim \exp\left\{i\frac{m}{2}\left(\frac{y_2-y_1}{\varepsilon}\right)^2\varepsilon\right\} = \exp\left\{iT_{kin}\varepsilon\right\}$$

$$\boxed{\langle x,t|x_0,t_0\rangle \sim \int \mathcal{D}x \, \exp i\int dt\,\mathcal{L} \sim \int \mathcal{D}x \, e^{iS}}$$

Übergangsamplitude = sum over all histories, weight e^{iS}



klass. Pfad = maxim. Gewicht durch Extremalwert von S=Wirkung

Funktionale: Zuordnung von Zahlen zu Funktionen

Integralfunktional: $F(\tilde{u}) = \int dx' \, f(x', \tilde{u}(x'))$

Ableitung: $\displaystyle \frac{\delta F(\tilde{u})}{\delta \tilde{u}(x)} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int dx' \left\{ f[x', \tilde{u}(x') + \varepsilon\,\delta(x-x')] - f(x', \tilde{u}(x'))\right\}$

$\displaystyle = \frac{\partial f}{\partial \tilde{u}}$ bei x Eigenschaften analog zu gewöhnlichen Ableitungen

Theorem: Green's Funktionen lassen sich durch Ableitung des Wirkungsfunktional [mit äußerer Quelle] berechnen.

Green's Funktion: $G(x_1,\dots,x_n) = {}_H\langle 0|T\{\varphi_H(x_1)\cdots\varphi_H(x_n)\}|0\rangle_H$ Heisenbergbild

$= \langle 0|T\{\varphi(x_1)\cdots\varphi(x_n)\, S\}|0\rangle$ WWbild, ohne Vakuumgraphen

bestimmen S-Matrix

Wirkungsfunktional: $W(j) = \langle 0|T \exp i\int d^4x\{\mathcal{L}_{int}(\varphi)+j\varphi\}|0\rangle$ WWbild

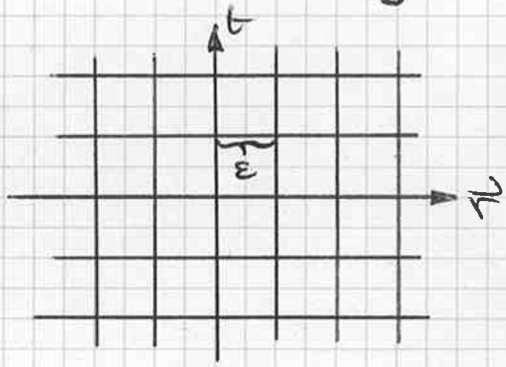
$$\Rightarrow i^n G(x_1,\dots,x_n) = \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)} W(j)/W(0) \quad \text{bei } j=0$$

freies Wirkungsfunktional: $W(j) = \exp i\int d^4x\, \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta j(x)}\right) W_0(j)$

$$W_0(j) = \langle 0|T\exp i\int d^4x\, j(x)\varphi(x)|0\rangle = \exp\left\{-\frac{i}{2}\int d^4x\, d^4y\, j(x)\Delta_F(x-y)j(y)\right\}$$

Pfadintegraldarstellung der Feldtheorie:

Minkowski-Raum zerlegt in Würfel α mit Kantenlänge $\varepsilon\to 0$; $\varphi_\alpha = \frac{1}{\varepsilon^4}\int d^4x\,\varphi(x)$



$$\int d^4x\, f(x) = \sum \varepsilon^4 f_\alpha \;;\quad \delta_4(x-y) = \frac{1}{\varepsilon^4}\delta_{\alpha\beta}$$

$$\partial\varphi(x) = \int d^4y\,[-\partial_y\delta_4(x-y)]\,\varphi(y) = \varepsilon^4\partial_{\alpha\beta}\varphi_\beta$$

$$= \frac{1}{2\varepsilon}[\varphi_{\alpha+1}-\varphi_{\alpha-1}] = \frac{1}{2\varepsilon}[\delta_{\alpha+1,\beta}-\delta_{\alpha-1,\beta}]\varphi_\beta$$

$\partial_{\alpha\beta} = \frac{1}{2\varepsilon^5}[\delta_{\alpha+1,\beta}-\delta_{\alpha-1,\beta}]$ schiefsymmetrisch

Definition Funktional $\mathcal{F}(\varphi)$ durch kontinuierl. Integral:

$$\boxed{\mathcal{F}(\varphi) = \lim_{\varepsilon\to 0}\int\prod_\alpha d\varphi_\alpha\, F\left(\sum_\beta\varepsilon^4 f(\varphi_\beta)\right) = \int\mathcal{D}\varphi\, F\left(\int d^4x\, f(\varphi(x))\right)}$$

Darstellung des freien Wirkungsfunktionals durch kontin. Integral

$$\tilde W_0(j) = \int\mathcal{D}\varphi\, \exp i\int d^4x\,(\mathcal{L}_0 + j\varphi)$$

$\mathcal{L}_0 = \frac{1}{2}\varphi(-\partial^2 - m^2 + i\varepsilon)\varphi$ freie Lagrangedichte

$$= \int \prod_\alpha d\varphi_\alpha \exp i\left[\varepsilon^8 \frac{1}{2} \varphi_\alpha K_{\alpha\beta} \varphi_\beta + \varepsilon^4 j_\alpha \varphi_\alpha\right]$$

trf. K auf Hauptdiagonale: $K = V^T K' V$ $\quad K' =$ diagonal

[K symm. $\Rightarrow$ V orthogonal] $\quad \varphi' = V\varphi$

$$\prod_\alpha d\varphi_\alpha = \prod_\alpha d\varphi'_\alpha \qquad [\mathrm{Det}\, V = 1]$$

$$\tilde{W}_0(j) = \int \prod_\alpha d\varphi'_\alpha \exp i\left[\varepsilon^8 \frac{1}{2} \varphi'_\alpha K'_{\alpha\alpha} \varphi'_\alpha + \varepsilon^4 (Vj)_\alpha \varphi'_\alpha\right]$$

Fresnel-Integral: $\displaystyle\int_{-\infty}^{\infty} dx\, e^{i[\rho x^2 + \sigma x]} = \sqrt{\frac{\pi i}{\rho}}\, e^{-i\frac{\sigma^2}{4\rho}}$

$$\tilde{W}_0(j) \sim \prod_\alpha \frac{1}{\sqrt{K'_{\alpha\alpha}}}\, e^{-\frac{i}{2}(jV^T)_\alpha K'^{-1}_{\alpha\alpha} (Vj)_\alpha} \sim \exp -\frac{i}{2} j_\alpha K^{-1}_{\alpha\beta} j_\beta \sim W_0(j)$$

$\quad \llcorner\, \mathrm{Det}\, K' = \mathrm{Det}\, K$

wenn: $\displaystyle\frac{1}{\varepsilon^8} K^{-1}_{\alpha\beta} = \Delta_F(x-y)$ $\quad$ Feynman-Propagator im Ortsraum

Auflösung:

$K_{\alpha\gamma} K^{-1}_{\gamma\beta} = \delta_{\alpha\beta}$ $\quad \Rightarrow \varepsilon^4 K_{\alpha\gamma} * \frac{1}{\varepsilon^8} K^{-1}_{\gamma\beta} = \frac{1}{\varepsilon^4}\delta_{\alpha\beta}$

$[-\partial^2 - m^2 + i\varepsilon]\, \Delta_F(x-y) = \delta_4(x-y)$ $\quad \Rightarrow \int d^4z \left\{[-\partial_x^2 - m^2 + i\varepsilon]\, \delta_4(x-z)\right\} \Delta_F(z-y) = \delta_4(x-y)$

Vergleich: $\quad K_{\alpha\beta} = [-\partial^2 - m^2 + i\varepsilon]\, \delta_4(x-y)$

Klein-Gordon Operator

Einführung der Gesamt-Lagrangedichte $\mathcal{L}$ durch Anwendung von

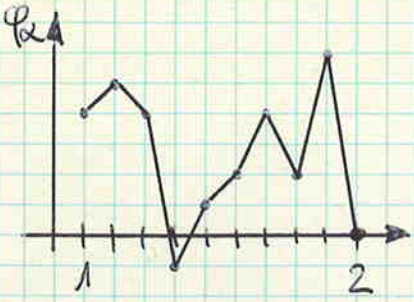
$$W(j) = \exp i \int d^4x\, \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta j(x)}\right) W_0(j)$$

$$\boxed{\begin{aligned} W(j) &\sim \int \mathcal{D}\varphi \exp i \int d^4x\, [\mathcal{L} + j\varphi] \\ &\sim \int \mathcal{D}\varphi \exp i\left[S + \int d^4x\, j\varphi\right]\end{aligned}}$$

Lösung der QFT auf Integration zurückgeführt

Propagator: $G(x_1,x_2) = \int \mathcal{D}\varphi\, \varphi(x_1)\varphi(x_2) \exp i \int d^4x\, \mathcal{L} \Big/ \int \mathcal{D}\varphi\, e^{iS}$

$= \int d\varphi_1 d\varphi_2\, \varphi_1 \varphi_2 \int \prod_{\alpha \neq 1,2} d\varphi_\alpha \exp i \varepsilon^4 \sum \mathcal{L}(\varphi_\beta) \Big/ \ldots$



Pfadintegral: 1.) störungstheor. lösbar durch Entwicklung nach Kopplung und nachfolgende Fresnel-Integration

2.) bei starker Kopplung numerische Integration von Integralen, die auf Raum-Zeit-Gittern definiert sind

FERMIONEN: Inkorporierung des Pauli-Prinzips

Grassmann-Variablen: η_i antikommut. c-Zahlen $\{\eta_i, \eta_j\} = 0$

$\eta_i^2 = 0$

Funktionen: Polynome $f(\eta) = a_0 + a_1 \eta$

$f(\eta_1, \eta_2) = a_0 + a_1\eta_1 + a_2\eta_2 + a_3\eta_1\eta_2$

$\vdots$

Differentiation: $\frac{\partial}{\partial \eta_i} \eta_j = \delta_{ij}$ $\quad \left\{\frac{\partial}{\partial \eta_i}, \eta_j\right\} = 0$ für $i \neq j$

$\left\{\frac{\partial}{\partial \eta_i}, \frac{\partial}{\partial \eta_j}\right\} = 0$

Integration: $z = \int\limits_{-\infty}^{\infty} dx\, f(x) = \int\limits_{-\infty}^{\infty} dx\, f(x+a)$ für gewöhnliche Zahlen

Grassmann-Variable: $z = \int d\eta\, f(\eta) = \int d\eta\, f(\eta + \xi)$

$= \int d\eta (f_0 + f_1 \eta) = \int d\eta (f_0 + f_1\eta + f_1 \xi) \Rightarrow$

$\int d\eta = 0$
$\int d\eta\, \eta = 1$

mehr-dim.: $\{d\eta_i, d\eta_j\} = 0$

$\{\eta_i, d\eta_j\} = 0$

(a) Integration ≡ Differentiation: $\int d\eta\, f(\eta) = \frac{\partial}{\partial\eta} f(\eta)$

(b) Variablentrf.: $I = \int d\eta_1 \cdots d\eta_n\, g(\eta) \qquad g(\eta) = \cdots + g_1 \eta_1 \cdots \eta_n$

$$= \pm g_1$$

$$\eta = M\xi \Longrightarrow \eta_1 \cdots \eta_n = \mathrm{Det}\, M\; \xi_1 \cdots \xi_n : \; g(\eta) \sim \mathrm{Det}\, M\; g(\xi)$$

$$\Big[\underline{\text{Bew}}: \eta_1\eta_2 = (M_{11}\xi_1 + M_{12}\xi_2)(M_{21}\xi_1 + M_{22}\xi_2)$$
$$= (M_{11}M_{22} - M_{12}M_{21})\, \xi_1\xi_2 \Big]$$

$$\Longrightarrow \underline{\underline{d\eta_1 \cdots d\eta_n = \mathrm{Det}\, M^{-1}\, d\xi_1 \cdots d\xi_n}}$$

<u>Grassmann-Felder</u>: $\eta_i \to \eta(x)$ kontinuierl. mit $\{\eta(x), \eta(y)\} = 0$

<u>Pfadintegral</u>: $W = \int \mathcal{D}\eta\, F\left(\int d^4x\, g(\eta(x))\right) = \int \prod_x d\eta_x\, F\left[\varepsilon^4 \sum_y g(\eta_y)\right]$

<u>Mathews-Salam-Formeln</u> $\int \mathcal{D}\bar\eta\, \mathcal{D}\eta\, e^{-\bar\eta Q \eta} \sim \mathrm{Det}\, Q$

$\int \mathcal{D}\bar\eta\, \mathcal{D}\eta\, \eta_i \bar\eta_j\, e^{-\bar\eta Q \eta} \sim Q^{-1}_{ij}\, \mathrm{Det}\, Q$ etc.

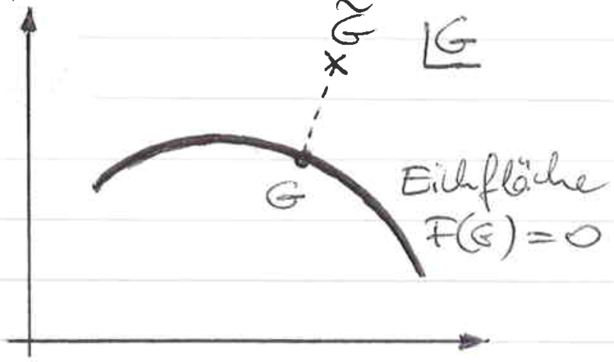
<u>Fermion-Wirkungsfunktional</u>: $\underline{\underline{W_{\bar\eta\eta} = \int \mathcal{D}\bar\psi\, \mathcal{D}\psi\, e^{i\int d^4x\, [\mathcal{L}(\bar\psi,\psi) + \bar\psi\eta + \bar\eta\psi]}}}$

Green's Fkt. durch Ableitung nach den Quellen $\eta(x)$ und $\bar\eta(x)$

§4. Eichung

<u>Wirkungsfunktional</u>: $W \sim \int \mathcal{D}G \exp i\int d^4x\, \mathcal{L}$

Integration über unendliche Bereiche, in denen $\mathcal{L}$ sich nicht ändert:



- ordne Integration so um, daß Integ. bereiche über physik. verschiedene und eichäquivalente Feldkonfigurationen getrennt
- faktorisiere unendliches Volumen ab: Feldbahnen nur über Eichfläche

<u>Eichfixierung</u>: $F(G)=0$

Anm: Für alle $\tilde{G}$ gibt es eindeutige Eichtrf. α so, daß

$$\tilde{G} \xrightarrow[\alpha]{} G \quad \text{mit } F(G)=0$$

Bsp: QED $\quad A_\mu = A'_\mu - \partial_\mu \Lambda \qquad F(A) = \partial_\mu A^\mu$

$$\partial_\mu A^\mu = 0 = \partial_\mu A'^\mu - \partial^2 \Lambda \Rightarrow \Lambda = \partial^{-2}(\partial A')$$

Integrationsumordnung: $\underline{\underline{\Delta(G) \int \mathcal{D}\tau_\alpha\, \delta(F(G^\alpha)) = 1}}$

ausführlich: $\Delta^{-1}(G) = \int \prod_x d\tau_{\alpha(x)} \prod_{x,a} \delta(F^a(G^\alpha_\mu))$

mit $d\tau_{\alpha(x)}$ = Hurwitz-Maß der $SU(3)$

$U(\alpha) = e^{-iT^a\alpha^a}$: $\quad U(\alpha')U(\alpha) = U(\alpha'\alpha)$

$$\Rightarrow U(\alpha'\alpha) = e^{-iT^a(\alpha'\alpha)^a} \quad \text{mit } (\alpha'\alpha)^a = \varphi^a(\alpha',\alpha)$$

dann: $d\tau_\alpha = J^{-1}(\alpha)\, d\alpha^1 \ldots d\alpha^8 \quad$ mit $J(\alpha) = \mathrm{Det}\left[\frac{\partial \varphi^a}{\partial \alpha'_b}\Big|_{\alpha'_b=0}\right]$

erfüllt: $d\tau_{\alpha'\cdot\alpha} = d\tau_{\alpha\cdot\alpha'} = d\tau_\alpha$ [eichinvariant]

$\Delta(G) = \Delta(G^\beta)$ ist eichinvariant:

$$\Delta^{-1}(G^\beta) = \int \mathcal{D}\tau_\alpha\, \delta\left(F(G^{\beta\alpha})\right) = \int \mathcal{D}\tau_{\beta\alpha}\, \delta\left(F(G^{\beta\alpha})\right) = \Delta^{-1}(G)$$

Wirkungsfunktional:

$$W \sim \int \mathcal{D}G\, \Delta(G) \int \mathcal{D}\tau_\alpha\, \delta\left(F(G^\alpha)\right) \exp i\int d^4x\, \mathcal{L}(G)$$

$$\underset{\text{(Eichinv.)}}{=} \int \mathcal{D}\tau_\alpha \int \underset{\substack{\uparrow \\ SU(3)_c\text{-invariant}}}{\mathcal{D}G^\alpha}\, \Delta(G^\alpha)\, \delta\left(F(G^\alpha)\right) \exp i\int d^4x\, \mathcal{L}(G^\alpha)$$

$$= \int \mathcal{D}\tau_\alpha * \int \mathcal{D}G\, \Delta(G)\, \delta\left(F(G)\right) \exp i\int d^4x\, \mathcal{L}(G)$$

$$\sim \int \mathcal{D}G\, \Delta(G)\, \delta\left(F(G)\right) \exp i\int d^4x\, \mathcal{L}(G)$$

Integration über Eichfläche und Eichorbit orthogonalisiert: $\int \mathcal{D}\tau_\alpha \sim \text{const}$ ist abtrennbar

infinitesimal: $F(G^\alpha(x)) = F(G(x)) + \int d^4y\, M_F(x,y)\, \alpha(y) + \ldots$

$$\Delta^{-1}(G) = \int \mathcal{D}\tau_\alpha\, \delta(M_F\alpha) \sim \int \mathcal{D}\alpha\, \delta(M_F\alpha) \sim \operatorname{Det} M_F^{-1}$$

$$\boxed{\begin{array}{l} \Delta(G) = \operatorname{Det} M_F \\ \text{Fadeev–Popov Determinante} \end{array}} \qquad M_F = \left.\frac{\delta F(G^\alpha)}{\delta\alpha}\right|_{\alpha=0}$$

BEISPIELE: $(G^\alpha)^a_\mu = G^a_\mu - f_{abc} G^b_\mu \alpha^c + \frac{1}{g_s}\partial_\mu \alpha^a + \mathcal{O}(\alpha^2)$

(i) Lorentz-Eichung: $\partial G = f$

$$\partial^\mu (G^\alpha)^a_\mu - f^a = \underbrace{(\partial^\mu G^a_\mu - f^a)}_{=0} \underbrace{- f_{abc}\partial^\mu G^b_\mu \alpha^c + \frac{1}{g_s}\partial^2 \delta_{ab}\alpha^b}$$

$$\frac{1}{g_s}\int d^4y \left\{\partial^2 \delta_{ab} + g_s f_{abc}\partial^\mu G^c_\mu(x)\right\}\delta_4(x-y)\alpha^b(y)$$

$$M_L^{ab}(x,y) = \frac{1}{g_s}\left[\partial^2 \delta_{ab} + g_s f_{abc}\partial^\mu G^c_\mu\right]\delta_4(x-y)$$

nicht-abel'sch: Det M_L manifest G abhängig
abel'sch, QED: Det M_L unabhängig von A $\longrightarrow$ ineffektiv

(ii) Axial-Eichung: $nG = 0$ $\qquad n^2 = \pm 1, 0$ [temporal/axial/Lichtkegel]

$$n(G^\alpha)^a = \underbrace{nG^a}_{=0} - f_{abc}\underbrace{nG^b}_{=0}\alpha^c + \frac{1}{g_s}n\partial\alpha^a$$

$$= \frac{1}{g_s}\int d^4y\, \delta_{ab}\, n\partial\, \delta_4(x-y)\,\alpha^b(y)$$

$$M_A^{ab}(x,y) = \frac{1}{g_s}n\partial\,\delta_{ab}\,\delta_4(x-y)$$

unabhängig von G $\longrightarrow$ ineffektiv

Effektive Lagrangedichte: $W \sim \int \mathcal{D}\text{Felder} \exp i\int d^4x\, \mathcal{L}_{eff}$

Lorentz-Eichung: phys. Resultate eichinvariant und unabhängig von f

$\Longrightarrow$ Mittelung über alle f mit Gewicht ρ $\Big\}$ $\rho(f) = \exp\frac{-i}{\xi}\int d^4x\, \mathrm{Tr}\, f^2$
ξ = freier Eichparameter

$$W \sim \int \mathcal{D}f\, \rho(f)\int \mathcal{D}G\, \delta(\partial G - f)\,\mathrm{Det}\, M_L \exp i\int d^4x\, \mathcal{L}$$

$$\sim \int \mathcal{D}G\, \mathrm{Det}\, M_L \exp i\int d^4x\,(\mathcal{L} + \mathcal{L}_{GF})$$

$\mathcal{L}_{GF} = \frac{-1}{\xi}\mathrm{Tr}(\partial G)^2$ Eichfixierungsterm

Geister: Grassmann-Oktettfelder

$$\mathrm{Det}\, M_L \sim \int \mathcal{D}\tilde{c}^* \mathcal{D}\tilde{c} \exp -i \int d^4x\, d^4y\, \tilde{c}_a^*(x) M_L^{ab} \tilde{c}_b(y)$$

$$(\tilde{c} = \sqrt{g_s}\, c) \nearrow \sim \int \mathcal{D}c^* \mathcal{D}c \exp i \int d^4x \left\{ (\partial^\mu c_a^*)(\partial_\mu c_a) + g_s f_{abc} (\partial^\mu c_a^*) G_\mu^c c_b \right\}$$

$$\sim \int \mathcal{D}c^* \mathcal{D}c \exp i \int d^4x\, \mathcal{L}_{FPG}$$

$$\uparrow$$

Geist-Lagrangedichte $= \partial c^* \mathcal{D} c$

Geistfelder: fermionische spinlose Hilfsfelder $\longrightarrow$
reell nicht existent [Spin-Statistik-Theorem];
treten nur in Loops gekoppelt an Gluonen auf.

Gesamt-Lagrangedichte der QCD:

$$W \sim \int \mathcal{D}\bar{q}\, \mathcal{D}q\, \mathcal{D}G\, \mathcal{D}c^* \mathcal{D}c \exp i \int d^4x\, \mathcal{L}_{eff}$$

$\mathcal{L}_{eff} = \mathcal{L}_{QCD} + \mathcal{L}_{GF} + \mathcal{L}_{FPG}$: $\mathcal{L}_{QCD}$ = Quark-Gluon-Lagrangedichte

$\mathcal{L}_{GF}$ = Eichfixierung

$\mathcal{L}_{FPG}$ = Geist-Lagrangedichte

Lorentz-Eichung:

$$\mathcal{L}_{GF} = \frac{-1}{\xi} \mathrm{Tr}(\partial G)^2$$

$$\mathcal{L}_{FPG} = \partial c^* (\partial + g_s f G) c$$

Axial-Eichung:

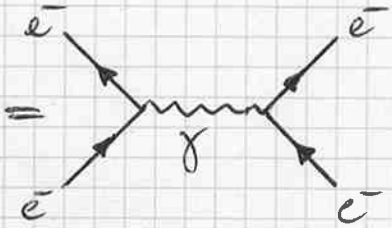
$$\mathcal{L}_{GF} = \frac{-1}{\xi} \mathrm{Tr}(nG)^2 \quad \text{für } \xi \to 0$$

$$\mathcal{L}_{FPG} = 0$$

§5. Asymptotische Freiheit

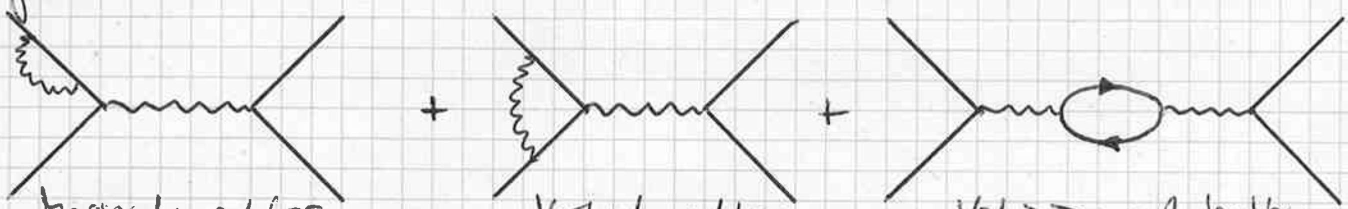
ERINNERUNG AN QED: $e^- e^-$ Streuung: $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 + \dots$

Bornterm:

$\mathcal{M}_0 =$ 

$= \frac{4\pi\alpha(\mu^2)}{Q^2} \dots \quad [Q^2 = -q^2]$

Strahlungskorrekturen:

$\mathcal{M}_1 =$ 

Massenkorrektur, Vertexkorrektur, Vakuumpolarisation

$+$ C.T.

Elemente [asymptotisch]

1.) Elektronpropagator: $G(p) = \frac{i}{\not p}\left\{1 + \xi \frac{\alpha}{4\pi} e_f^2 \log\left(\frac{-p^2}{\mu^2}\right)\right\}$

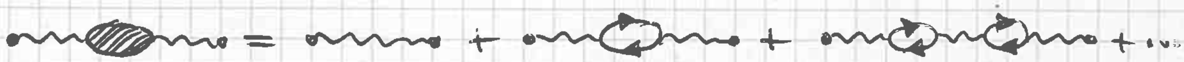
2.) Vertex: $\Gamma_\mu = -i e e_f \gamma_\mu \left\{1 - \xi \frac{\alpha}{4\pi} e_f^2 \log\frac{Q^2}{\mu^2} + \dots\right\}$

IR-Anteile

3.) Photonpropagator: $\Pi_{\mu\nu} = -i\frac{\alpha}{3\pi}\sum_f e_f^2 \left[q_\mu q_\nu - q^2 g_{\mu\nu}\right] \log\frac{Q^2}{\mu^2}$

$$\Longrightarrow \mathcal{M} = \mathcal{M}_0\left[1 + \frac{\alpha(\mu^2)}{3\pi}\sum_f e_f^2 \log\frac{Q^2}{\mu^2} + \dots\right] \equiv \frac{4\pi\alpha(Q^2)}{Q^2}$$

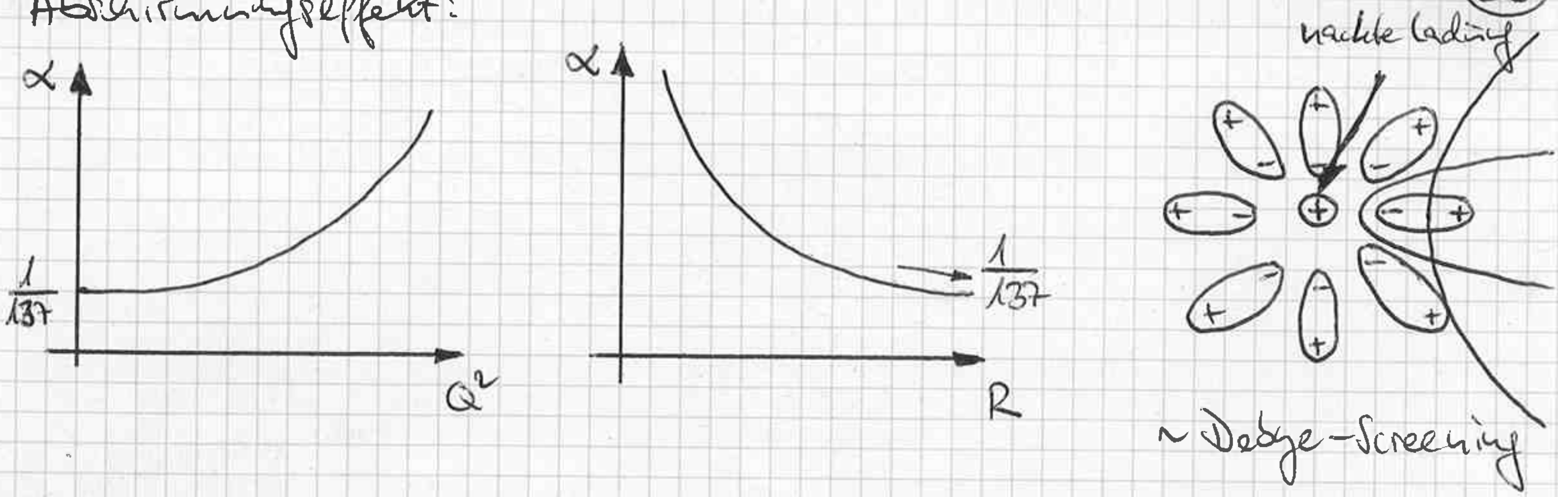
$$\Longrightarrow \alpha(Q^2) = \alpha(\mu^2)\left[1 + \sum_f e_f^2 \frac{\alpha(\mu^2)}{3\pi} \log\frac{Q^2}{\mu^2}\right]$$

Summation: 

effektive elektrische Ladung:

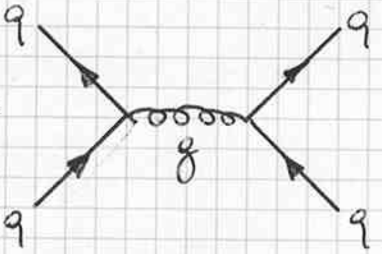
$$\boxed{\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \sum_f e_f^2 \frac{\alpha(\mu^2)}{3\pi} \log\frac{Q^2}{\mu^2}}}$$

Abschirmungseffekt:



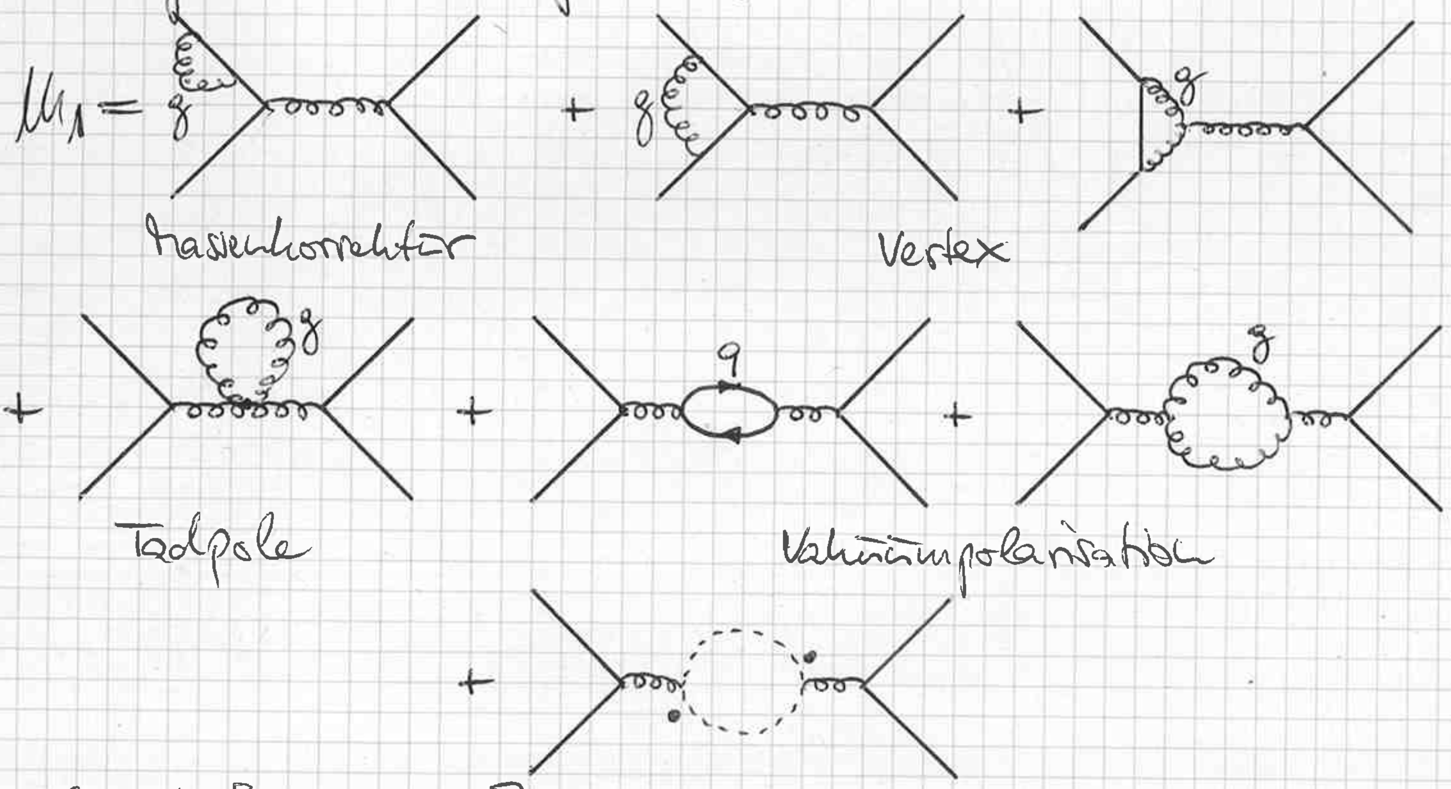
Übertragung auf QCD: Quark-Quark-Streuung $\mathcal{M}=\mathcal{M}_0+\mathcal{M}_1+\dots$

Bornterms

$\mathcal{M}_0 =$ 

$= \frac{4\pi\alpha_s(\mu^2)}{Q^2}\dots$

Strahlungskorrekturen [generisch]

$\mathcal{M}_1 =$ 

Elemente [asymptotisch]:

1.) Quarkpropagator: $e^2 \longrightarrow g_s^2\,(T^aT^a)_{ij} = \frac{4}{3}\, g_s^2\,\delta_{ij} = \frac{N^2-1}{2N}\, g_s^2\,\delta_{ij}$

$$G_{ij} = i\,\frac{\delta_{ij}}{\not p}\left\{1+\xi\,\frac{\alpha_s}{4\pi}\,\frac{N^2-1}{2N}\,\log\left(\frac{-p^2}{\mu^2}\right)\right\}$$

2.) Vertex:

$$\Gamma^a_{\mu, ij} = T^a_{ij}\, g_S \gamma^\mu \left\{ 1 - \frac{\alpha_s}{4\pi} \log\frac{Q^2}{\mu^2}\left[\xi \frac{N^2-1}{2N} + \left(1 - \frac{1-\xi}{4}\right) N\right] + (IR) \right\}$$

3.) Gluonpropagator: Tadpole $= 0$

Fermionloop: $e^2 \to g_s^2\, \mathrm{Tr}(T^a T^b) = \frac{1}{2} g_s^2 \delta^{ab}$

$$I^q_{\mu\nu} = -i \frac{\alpha_s}{3\pi} N_F \left[q_\mu q_\nu - q^2 g_{\mu\nu}\right] \frac{\delta^{ab}}{2} \log\frac{Q^2}{\mu^2}$$

Gluonloop: $$I^g_{\mu\nu} = i \frac{\alpha_s}{4\pi} N \delta^{ab}\left[\frac{11}{6} q_\mu q_\nu - \frac{19}{12} q^2 g_{\mu\nu} + \frac{1-\xi}{2}\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\right] * \log\frac{Q^2}{\mu^2}$$

nicht transversal / eichabhängig

Geistloop: $$I^G_{\mu\nu} = -i \frac{\alpha_s}{4\pi} N \delta^{ab}\left[\frac{1}{6} q_\mu q_\nu + \frac{1}{12} q^2 g_{\mu\nu}\right] \log\frac{Q^2}{\mu^2}$$

$$\Longrightarrow I^g_{\mu\nu} + I^G_{\mu\nu} = i \frac{\alpha_s}{4\pi} N \delta^{ab}\left(\frac{5}{3} + \frac{1-\xi}{2}\right)\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right) \log\frac{Q^2}{\mu^2}$$

Geister transversalisieren Gluonloop, aber eichabhängig

$$i\frac{-g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2}}{q^2} \to i\frac{-g_{\mu\rho} + (1-\xi)\frac{q_\mu q_\rho}{q^2}}{q^2}\, I^{\rho\sigma}\, i\frac{-g_{\sigma\nu} + (1-\xi)\frac{q_\sigma q_\nu}{q^2}}{q^2}$$

Addition aller Terme:

$$\mathcal{M}(qq \to qq) = \mathcal{M}_0 \left\{ 1 + \frac{\alpha_s}{4\pi}\left[\underset{\text{q-loop}}{\frac{2}{3} N_F} - \underset{\text{g-loop}}{\frac{13}{6} N} - \underset{\text{g-Vertex}}{\frac{3}{2} N}\right] \log\frac{Q^2}{\mu^2} + (IR) \right\}$$

$$= \frac{4\pi\alpha_s(Q^2)}{Q^2} \cdots$$

$$\Longrightarrow \alpha_s(Q^2) = \alpha_s(\mu^2)\left[1 - \frac{11N - 2N_F}{12} \frac{\alpha_s(\mu^2)}{\pi} \log\frac{Q^2}{\mu^2}\right] + O(\alpha_s^3)$$

Resummation:

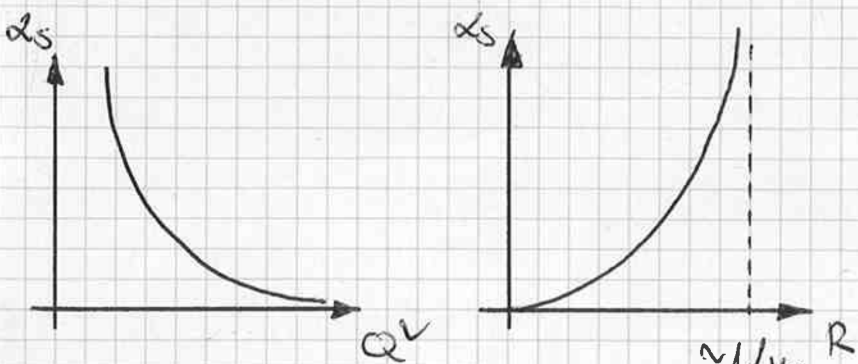
$$\boxed{\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{33-2N_F}{12}\frac{\alpha_s(\mu^2)}{\pi}\log\frac{Q^2}{\mu^2}}}$$

Mit wachsendem Q^2 verschwindet die effektive Colorladung:
<u>asymptotische Freiheit</u>

[nicht-abel'sche $SU(3)$: $N_F \leq 16$]

Folge nicht-abel'scher Eichbosonloops; konträr zu $U(1)$ [und <u>allen</u> anderen Theorien]

[Politzer '73, Gross + Wilczek '73, 't Hooft ?]



$\sim$ Confinement-Radius

<u>Skalenparameter der QCD</u>: Quantentheorie führt in unskalierte klassische Chromodynamik [für $m_q = 0$] eine Skala via Renormierung ein: Vorgabe einer Kopplungskonstanten für vorgegebenen Abstand:

$$\alpha_s = \alpha_s(\mu^2)$$

$\llcorner$ experimentell bestimmt

Reformulierung:
$$\frac{1}{\alpha_s(Q^2)} = \underbrace{\frac{1}{\alpha_s(\mu^2)} - \frac{33-2N_F}{12\pi}\log\mu^2}_{\equiv \frac{33-2N_F}{12\pi}\log\frac{1}{\Lambda^2}} + \frac{33-2N_F}{12\pi}\log Q^2$$

$$\Longrightarrow \boxed{\alpha_s(Q^2) = \frac{12\pi}{(33-2N_F)\log\frac{Q^2}{\Lambda^2}}}$$

$\Lambda^{-1} \sim 1\,\mathrm{fm} \sim$ Confinement-Radius

$\Longrightarrow \boxed{\Lambda \sim 100 - 300\ \mathrm{MeV}}$

$\frac{\alpha_s(Q^2)}{\pi} \lesssim 10^{-1}$ für $Q^2 \gtrsim 2\,\mathrm{GeV}^2 \Longrightarrow$ Bereich gesicherter Störungstheorie

Renormierungsgruppengleichung:

$$\mu^2 \frac{\partial \alpha_s(\mu^2)}{\partial \mu^2} = \beta\left(\alpha_s(\mu^2)\right) \qquad \beta(\alpha_s) = -\beta_0 \frac{\alpha_s^2}{\pi} + O(\alpha_s^3)$$

Lösung: $$\log \frac{Q^2}{\mu^2} = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{\beta(\alpha_s)} = -\frac{\pi}{\beta_0}\left[\frac{1}{\alpha_s(\mu^2)} - \frac{1}{\alpha_s(Q^2)}\right]$$

$$\boxed{\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+\beta_0 \frac{\alpha_s(\mu^2)}{\pi} \log \frac{Q^2}{\mu^2}} \qquad \beta_0 = \frac{33-2N_F}{12}}$$

RGG bestimmt asymptot. Verhalten der laufenden Kopplung

Höhere Ordnungen: $$\beta(\alpha_s) = -\frac{\alpha_s^2}{\pi}\left[\beta_0 + \beta_1 \frac{\alpha_s}{\pi} + \beta_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \ldots\right]$$

$$\beta_1 = \frac{153-19N_F}{24} \qquad \beta_2 = \frac{1}{128}\left\{2857 - \frac{5033}{9} N_F + \frac{325}{27} N_F^2\right\}$$

$$\alpha_s(Q^2) = \frac{\pi}{\beta_0 \log \frac{Q^2}{\Lambda^2}}\left\{1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log \frac{Q^2}{\Lambda^2}}{\log \frac{Q^2}{\Lambda^2}} + \ldots\right\}$$

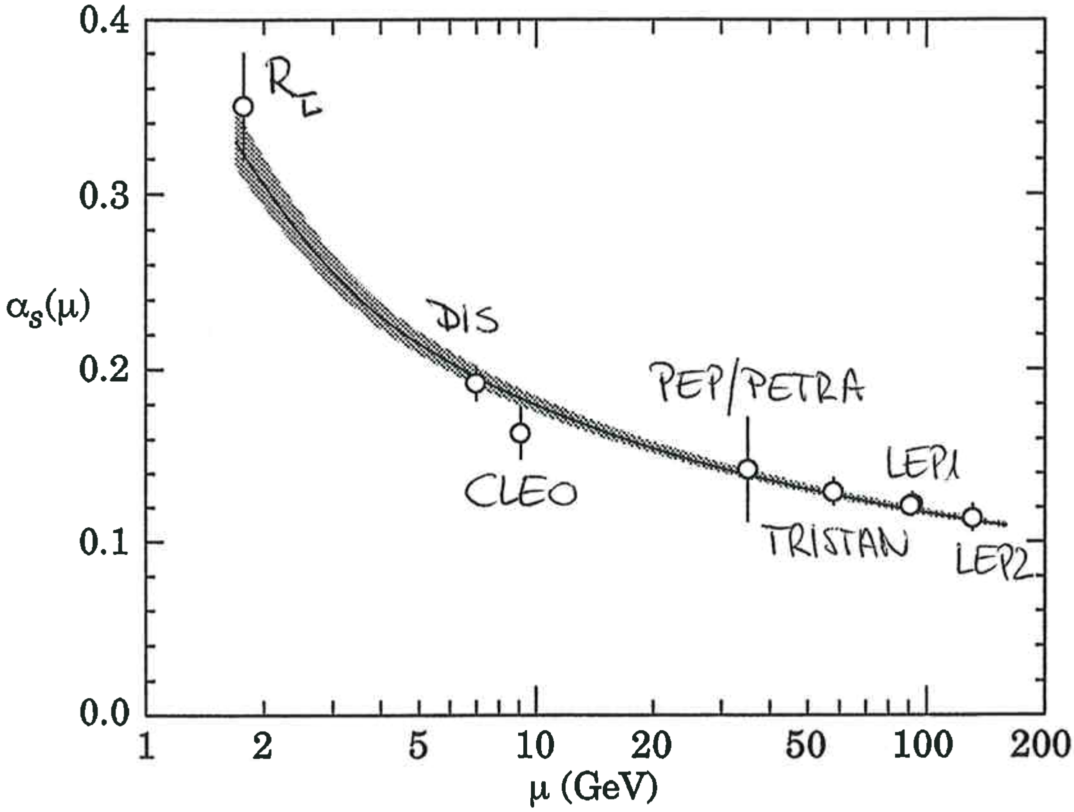
RENORMIERUNGSSCHEMATA $[n=4-2\varepsilon]$

Fermionpropagator: $S^{-1}(p) = \not{p}\left[1-\tilde{\Sigma}(p)\right]$

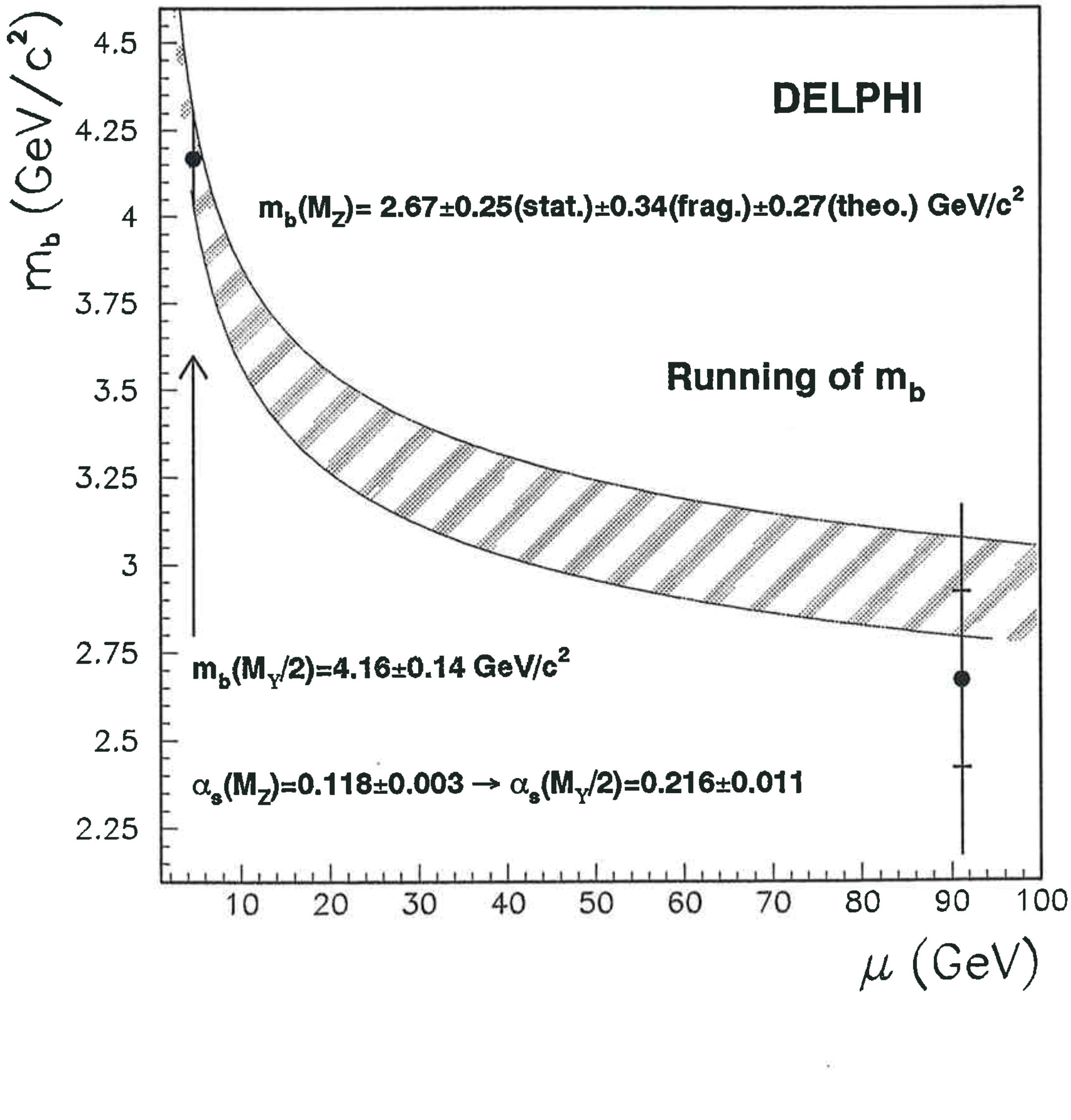
$$\tilde{\Sigma}(p) = \frac{4}{3} \frac{g_s^2}{(4\pi)^{2-\varepsilon}} (\mu f)^{2\varepsilon} \frac{\Gamma(\varepsilon)}{(-p^2)^\varepsilon} 2(1-\varepsilon)\, B(2-\varepsilon, 1-\varepsilon)$$

$$g_s^2 \longrightarrow g_s^2 (\mu f)^{2\varepsilon}$$

$(\mu f)^{2\varepsilon}$ [f = beliebige Konstante] macht Wirkung dimensionslos!



0.4
0.3
0.2
0.1
0.0
$\alpha_s(\mu)$
1
2
5
10
20
50
100
200
μ (GeV)
R_τ
DIS
CLEO
PEP/PETRA
TRISTAN
LEP1
LEP2



DELPHI
$m_b(M_Z)$= 2.67±0.25(stat.)±0.34(frag.)±0.27(theo.) GeV/c^2
Running of m_b
$m_b(M_\Upsilon/2)$=4.16±0.14 GeV/c^2
$\alpha_s(M_Z)$=0.118±0.003 → $\alpha_s(M_\Upsilon/2)$=0.216±0.011
m_b (GeV/c^2)
4.5
4.25
4
3.75
3.5
3.25
3
2.75
2.5
2.25
10
20
30
40
50
60
70
80
90
100
μ (GeV)

$$S^{-1}(p) = p\left\{1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\left[\frac{1}{\varepsilon} - \log\frac{-p^2}{(\mu f)^2} + 1 + \log 4\pi - \gamma_E\right]\right\}$$

Euler'sche Konstante

$[\Gamma(x) = \frac{1}{x} - \gamma_E + O(x)]$

multiplikative Renormierung:

$$S^{-1}(p) = Z_4^{-1} S_R^{-1}(p)$$

(i) Dyson'sches Renormierungsschema

fordere: $f = 1$

$S_R^{-1} = p$ für $\mu^2 = -p^2$

$$\left.\begin{matrix} f=1 \\ S_R^{-1} = p \text{ für } \mu^2=-p^2 \end{matrix}\right\} S^{-1}(p) = p\left[1 - \tilde{\Sigma}(\mu)\right]\left[1 - \tilde{\Sigma}(p) + \tilde{\Sigma}(\mu)\right]$$

Lösung: $Z_4^{-1} = 1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\left[\frac{1}{\varepsilon} + \log 4\pi - \gamma_E + 1\right]$

$$S_R^{-1}(p) = p\left[1 + \frac{4}{3}\frac{g_{s\,\mathrm{MOM}}^2}{16\pi^2}\log\left(\frac{-p^2}{\mu^2}\right)\right]$$

(MOM = momentum subtraction)

$\boxed{\text{Kopplung / Ladung hängt vom Renormierungsschema ab}}$

(ii) 't Hooft: Minimal subtraction (MS)

fordere: $f = 1$

Z_4^{-1} entfernt nur $\frac{1}{\varepsilon}$-Pol

$$S^{-1}(p) = p\left[1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\frac{1}{\varepsilon}\right]\left\{1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\left[-\log\left(\frac{-p^2}{\mu^2}\right) + \log 4\pi - \gamma_E + 1\right]\right\}$$

Lösung: $Z_4^{-1} = 1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\frac{1}{\varepsilon}$

$$S_R^{-1}(p) = p\left\{1 - \frac{4}{3}\frac{g_{s\,\mathrm{MS}}^2}{16\pi^2}\left[-\log\left(\frac{-p^2}{\mu^2}\right) + \log 4\pi - \gamma_E + 1\right]\right\}$$

(iii) Modifizierte Minimalsubtraktion ($\overline{\mathrm{MS}}$) Bardeen, ...

fordere: $f = \exp\left[-\frac{1}{2}(\log 4\pi - \gamma_E)\right]$ ← entfernt triviale Konstanten

$$S^{-1}(p) = p\left[1-\frac{4}{3}\frac{g_s^2}{16\pi^2}\frac{1}{\varepsilon}\right]\left[1-\frac{4}{3}\frac{g_s^2}{16\pi^2}\left(1-\log\frac{-p^2}{\mu^2}\right)\right]$$

Lösung: $Z_\psi^{-1} = 1-\frac{4}{3}\frac{g_s^2}{16\pi^2}\frac{1}{\varepsilon}$

$$S_R^{-1}(p) = p\left\{1-\frac{4}{3}\frac{g_{s\overline{MS}}^2}{16\pi^2}\left[1-\log\left(\frac{-p^2}{\mu^2}\right)\right]\right\}$$

$\underline{MS \longleftrightarrow \overline{MS}}$: $\mu^2 \longleftrightarrow \mu^2 \exp\left[-\log 4\pi + \gamma_E\right]$

$$\Lambda_{MS}^2 = \mu^2 \exp\left\{-\frac{16\pi^2}{\beta_0 g_{sMS}^2} + \frac{\beta_1}{\beta_0^2}\log\left(\beta_0 g_{sMS}^2\right)\right\}$$

$$\Lambda_{\overline{MS}}^2 = \mu^2 \exp\left\{-\frac{16\pi^2}{\beta_0 g_{s\overline{MS}}^2} + \frac{\beta_1}{\beta_0^2}\log\left(\beta_0 g_{s\overline{MS}}^2\right)\right\}$$

$$\boxed{\Lambda_{\overline{MS}} = \Lambda_{MS} \exp\left\{\frac{\log 4\pi - \gamma_E}{2}\right\}}$$

β_0, β_1 unabhängig vom Renormierungsschema (nicht $\beta_{i\geq 2}$)

$\alpha_{s\overline{MS}}(Q^2) > \alpha_{sMS}(Q^2)$

Quarkmassen

Quark-Selbstenergie:

$$\Sigma(p=m) = m\, C_F \frac{\alpha_s}{\pi}\Gamma(1+\varepsilon)\left(\frac{4\pi\mu_0^2}{m^2}\right)^{\varepsilon}\left(\frac{3}{4\varepsilon}+1\right)$$

$m = m_0 + \Sigma(p=m)$ Polmasse

$\overline{m}(\mu^2) = m_0 + \delta\overline{m}$ $\overline{MS}$-Masse

$\delta\overline{m} = m\, C_F \frac{\alpha_s}{\pi}\Gamma(1+\varepsilon)\left(\frac{4\pi\mu_0^2}{\mu^2}\right)^{\varepsilon}\frac{3}{4\varepsilon}$ [nur Divergenz]

25e

Relation Polmasse $\Longleftrightarrow$ $\overline{MS}$-Masse:

$$\overline{m}(\mu^2) = m - \left[\Sigma(\not p = m) - \delta\overline{m}\right] = m\left[1 - C_F\frac{\alpha_s}{4}\left(\frac{3}{4}\log\frac{\mu^2}{m^2} + 1\right)\right]$$

$$= m\left[1 - C_F\frac{\alpha_s}{4}\right]\left[1 - \frac{3}{4}C_F\frac{\alpha_s}{4}\log\frac{\mu^2}{m^2}\right]$$

$$\boxed{\begin{aligned}\overline{m}(m^2) &= m\left[1 - C_F\frac{\alpha_s(m^2)}{4}\right]\\ \overline{m}(\mu^2) &= \overline{m}(m^2)\left[1 - \frac{\alpha_s}{4}\log\frac{\mu^2}{m^2}\right]\end{aligned}}$$

Renormierungsgruppengleichung:

$$\mu^2\frac{\partial\overline{m}(\mu^2)}{\partial\mu^2} = -\gamma_m(\alpha_s(\mu^2))\,\overline{m}(\mu^2)$$

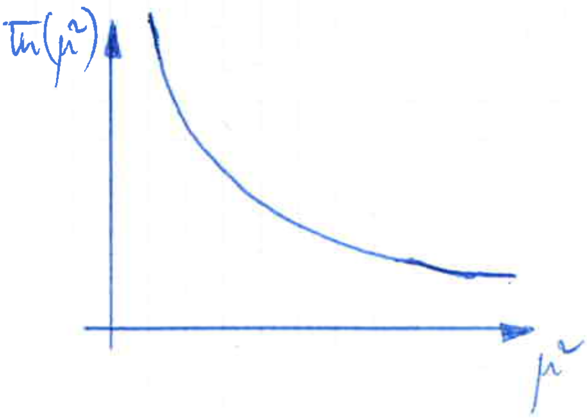
$$\gamma_m(\alpha_s) = \frac{\alpha_s}{4} + O(\alpha_s^2)$$ anomale Massendimension

$$\alpha_s(\mu^2) = \frac{4}{\beta_0\log\frac{\mu^2}{\Lambda^2}} \Longrightarrow$$ Lösung: $$\overline{m}(\mu^2) = \overline{m}(m^2)\exp\left\{\frac{-1}{\beta_0}\int_{m^2}^{\mu^2}\frac{dQ^2}{Q^2\log\frac{Q^2}{\Lambda^2}}\right\}$$

$$= \overline{m}(m^2)\left[\frac{\alpha_s(\mu^2)}{\alpha_s(m^2)}\right]^{\frac{1}{\beta_0}}$$

$$\boxed{\begin{aligned}\overline{m}(\mu^2) &= \hat{m}\left[\alpha_s(\mu^2)\right]^{\frac{1}{\beta_0}}\\ \hat{m} &= \overline{m}(m^2)\left[\alpha_s(m^2)\right]^{-\frac{1}{\beta_0}}\end{aligned}}$$

[RG-Invariante]



Mit wachsendem μ^2 ($R \to 0$) verschwindet die effektive Quarkmasse.

Beispiele:

Bottomquark: $m_b = 4.8$ GeV $\qquad \overline{m}_b(m_b^2) = 4.2$ GeV

$\overline{m}_b(M_Z^2) = 2.9$ GeV

Charmquark: $m_c = 1.6$ GeV $\qquad \overline{m}_c(m_c^2) = 1.2$ GeV

$\overline{m}_c(M_Z^2) = 0.6$ GeV

leichte Quarks: $\overline{m}_u(1\,\mathrm{GeV}^2) \sim 5$ MeV $\qquad$ Gasser, Leutwyler

[QCD sum rules] $\overline{m}_d(1\,\mathrm{GeV}^2) \sim 8$ MeV

$\overline{m}_s(1\,\mathrm{GeV}^2) \sim 200$ MeV

Höhere Ordnungen:

$$\overline{m}(m^2) = \frac{m}{1 + C_F \frac{\alpha_s(m^2)}{\pi} + K\left(\frac{\alpha_s(m^2)}{\pi}\right)^2}$$

Gray, Broadhurst, Grafe, Schilcher

$K_t \sim 10.9 \qquad K_b \sim 12.4 \qquad K_c \sim 13.4$

$$\overline{m}(\mu^2) = \overline{m}(m^2)\,\frac{c[\alpha_s(\mu^2)/\pi]}{c[\alpha_s(m^2)/\pi]}$$

$$c(x) = \left(\tfrac{9}{2}x\right)^{4/9}\left[1 + 0.895x + 1.371x^2 + 1.952x^3\right] \qquad m_s < \mu < m_c$$

$$c(x) = \left(\tfrac{25}{6}x\right)^{12/25}\left[1 + 1.014x + 1.389x^2 + 1.091x^3\right] \qquad m_c < \mu < m_b$$

$$c(x) = \left(\tfrac{23}{6}x\right)^{12/23}\left[1 + 1.175x + 1.501x^2 + 0.1725x^3\right] \qquad m_b < \mu < m_t$$

$$c(x) = \left(\tfrac{7}{2}x\right)^{4/7}\left[1 + 1.389x + 1.793x^2 - 0.6834x^3\right] \qquad m_t < \mu$$

Chetyrkin

Larin, van Ritbergen, Vermaseren

§6. Renormierungsgruppe

Parameter einer Feldtheorie [Massen, Kopplungen] werden für ein bestimmtes μ^2 vorgegeben; die physikalischen Observablen sind unabhängig vom speziellen μ^2:

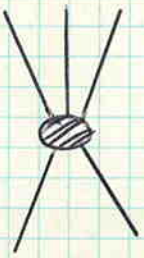
Änderung von μ^2 $\oplus$ entsprechende Änderung der Parameter

$\Longrightarrow$ Invarianz, formuliert durch RGGn [$\Leftarrow$ partielle Differentialglgn.]

Anwendung: μ^2-Variation wird überschoben auf Q^2-Variation mit Hilfe von Dimensionsanalyse

$\Longrightarrow$ <u>Q^2-Variation von Observablen bestimmt</u>

Ableitung der RGG:

Green's-Funktion: $G^{N_G N_\psi}(p) =$ 

$N_G = \#$ Eichfelder

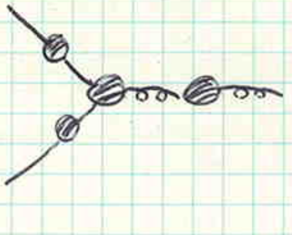
$N_\psi = \#$ Fermionfelder

$$= \langle 0|T\{\psi(x_1)\cdots\}|0\rangle_{FT}$$

amputierte Green's-Funktionen:

$$\Gamma^{N_G N_\psi}(p) = \frac{G^{N_G N_\psi}(p)}{\prod_G G^{2,0}(p_G) \prod_\psi G^{0,2}(p_\psi)}$$

<u>Bsp:</u> $\Gamma^{0,2} = \left[G^{0,2}\right]^{-1}$

$G^{1,2} =$  $\Longrightarrow \Gamma^{1,2} =$  $\left[= g_{s0} T^a \gamma_\mu + \cdots\right]$

Vertex

Theorem der multiplikativen Renormierung von Eichtheorien

Divergente Teile der Γ's können in cut-off-abhängige Faktoren abgespalten werden; der verbleibende Rest Γ_R ist nach Einführung der renormierten Ladung g endlich und wohldefiniert für cut-off $\to \infty$; die Renormierungskonstanten hängen nur von den Spezies der äußeren Beine ab.

Bsp: (i) Fermionpropagator:

$i S_F'(p) =$  $+ \dots$

$$= \frac{i}{\not p} + \frac{i}{\not p}\left[-i\Sigma(p)\right]\frac{i}{\not p} + \dots$$

$$\Sigma(p,\varepsilon) = -C_F \frac{\alpha_{s0}}{4\pi}\Gamma(1+\varepsilon)\left(\frac{4\pi}{-p^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}+1+O(\varepsilon)\right)\not p$$

$$\frac{i}{\not p} \longrightarrow \frac{i}{\not p}\left[1-\frac{\alpha_{s0}}{4\pi}C_F\Gamma(1+\varepsilon)\left(\frac{4\pi}{\mu^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}+1\right)-C_F\frac{\alpha_{s0}}{4\pi}\log\frac{\mu^2}{-p^2}\right] + O(\varepsilon)$$

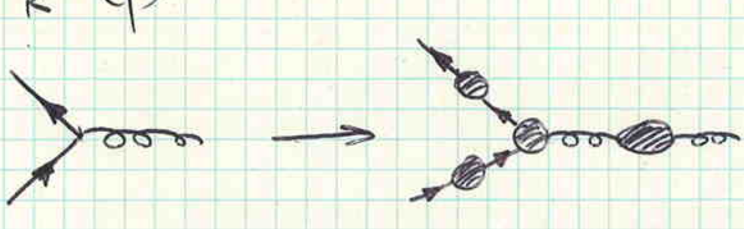
$$= \frac{i}{\not p}\left[1-C_F\frac{\alpha_{s0}}{4\pi}\Gamma(1+\varepsilon)\left(\frac{4\pi}{\mu^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}+1\right)\right]\left[1-C_F\frac{\alpha_{s0}}{4\pi}\log\frac{\mu^2}{-p^2}\right]+O(\varepsilon,\alpha_s^2)$$

$$S_F'(p) = \frac{Z_\psi(\alpha_{s0},\mu)}{\not p}\left[1-C_F\frac{\alpha_{s0}}{4\pi}\log\frac{\mu^2}{-p^2}\right]$$

$$\longleftarrow S_F^R(p) = \frac{1}{\not p} \text{ für } \mu^2 = -p^2$$

$$\Gamma^{q,2} = Z_\psi^{-2/2}\,\Gamma_R^{q2}(p)$$

(ii) Vertex:



$$S_F(p')g_{s0}T^a\gamma_\mu S_F(p)D_G^{\mu\nu}(k) \longrightarrow S_F'(p')g_{s0}T^a\Gamma_\mu' S_F'(p)D_G'^{\mu\nu}(k)$$

$$= Z_\psi^{1/2}S_F^R(p')\left[g_{s0}\frac{Z_\psi Z_G^{1/2}}{Z_1}\right]\Gamma_\mu^R S_F^R(p)Z_\psi^{1/2}D_G^{R\mu\nu}(k)Z_G^{1/2}T^a$$

$$= Z_\psi^{-1/2}S_F'(p')\underbrace{\left[g_{s0}\frac{Z_\psi Z_G^{1/2}}{Z_1}\right]}_{g_s}\Gamma_\mu^R S_F'(p)Z_\psi^{-1/2}D_G'^{\mu\nu}(k)Z_G^{-1/2}T^a$$

$$\Longrightarrow g_{s0}\,\Gamma'_\mu = Z_\psi^{-2/2}\, Z_G^{-1/2}\, g_s \Gamma^R_\mu$$

$$\boxed{\Gamma^{N_G N_\psi}(p_i\, g_{s0}, \varepsilon) = Z_G^{-N_G/2}(g_{s0}, \mu)\, Z_\psi^{-N_\psi/2}(g_{s0}, \mu)\, \Gamma_R^{N_G N_\psi}(p_i\, g_s, \mu)}$$

In Eichtheorien werden Renormierungskonstanten und g_s durch 3 Green's-Funktionen theoretisch fixiert [mod. Eichung / Geister]:

$$\Gamma_R^{20}(p^2=-\mu^2) = Z_G(g_{s0}, \mu)\, \Gamma^{20}(p^2=-\mu^2) = -g_{\mu\nu} p^2 + p_\mu p_\nu$$

$$\Gamma_R^{0,2}(p^2=-\mu^2) = Z_\psi(g_{s0}, \mu)\, \Gamma^{02}(p^2=-\mu^2) = \not p$$

$$\Gamma_R^{21}(p^2=-\mu^2) = \sqrt{Z_G}\, Z_\psi\, \Gamma^{21}(p^2=-\mu^2) = g_s \gamma_\mu$$

<u>RGG:</u> Die synchrone Änderung von μ und $g_s(\mu)$ läßt die Theorie invariant.

$$\mu \frac{d}{d\mu} \Gamma = 0$$

$$\Longrightarrow \left\{ \mu \frac{\partial}{\partial\mu} + \mu \frac{\partial g_s}{\partial\mu}\frac{\partial}{\partial g_s} - \frac{N_G}{2}\mu \frac{\partial \log Z_G}{\partial\mu} - \frac{N_\psi}{2}\frac{\partial \log Z_\psi}{\partial\mu} \right\} \Gamma_R^{N_G N_\psi}(p_i\, g_s(\mu), \mu) = 0$$

β-Funktion: $\beta(g_s) = \mu \frac{\partial}{\partial\mu} g_s(g_{s0}, \mu)$

anomale Dimension: $\gamma(g_s) = \frac{1}{2}\mu \frac{\partial}{\partial\mu} \log Z(g_{s0}, \mu)$

$$\boxed{\left\{ \mu \frac{\partial}{\partial\mu} + \beta(g_s)\frac{\partial}{\partial g_s} - N_G \gamma_G(g_s) - N_\psi \gamma_\psi(g_s) \right\} \Gamma_R(p_i\, g_s, \mu) = 0}$$

Überschiebe μ-Variation auf p-Variation: $\Gamma_R = \mu^D f\left(\frac{p}{\mu}\right)$

D = Ingenieur-Dimension

$p \to e^t p$:

$$\boxed{\left\{-\frac{\partial}{\partial t} + \beta(g_s)\frac{\partial}{\partial g_s} + D - N_G\gamma_G(g_s) - N_\psi\gamma_\psi(g_s)\right\}\Gamma_R^{N_G,N_\psi}(e^t p, g_s, \mu) = 0}$$

wobei $t = \log\frac{Q}{\mu}$

Lösung:
$$\boxed{\begin{aligned}\frac{\partial \bar{g}_s(g_s,t)}{\partial t} &= \beta(\bar{g}_s(g_s,t))\\ \bar{g}_s(g_s,0) &= g_s\end{aligned}} \Longrightarrow \boxed{t = \int_{g_s}^{\bar{g}_s(g_s,t)} \frac{dg'}{\beta(g')}}$$

- Differentiation nach t: $1 = \frac{1}{\beta(\bar{g}_s)}\frac{\partial \bar{g}_s}{\partial t}$
- Differentiation nach g_s:

$$0 = -\frac{1}{\beta(g_s)} + \frac{1}{\beta(\bar{g}_s)}\frac{\partial \bar{g}_s}{\partial g_s} \longrightarrow \beta(g_s)\frac{\partial \bar{g}_s}{\partial g_s} = \beta(\bar{g}_s) = \frac{\partial \bar{g}_s}{\partial t}$$

Die allgemeinste Lösung ist eine Funktion von $\bar{g}_s(g_s,t)$, modifiziert um die durch Ingenieur-Dimension und anomale Dimension bestimmte spezielle Lösung:

$$\boxed{\Gamma_R^{N_G,N_\psi}(e^t p, g_s) = \Gamma_R^{N_G,N_\psi}(p, \bar{g}_s(g_s,t)) \exp\left\{Dt - \int_0^t dt' \left[N_G\gamma_G(\bar{g}_s(g_s,t')) + N_\psi\gamma_\psi(\bar{g}_s(g_s,t'))\right]\right\}}$$

$\gamma_G(g_s) = \left(-\frac{13}{2} + \frac{2}{3}N_F\right)\frac{\alpha_s}{4\pi} + \ldots$ [Landau-Eichung]

$\gamma_\psi(g_s) = 0 + \ldots$

$\frac{\beta(g_s)}{g_s} = -\beta_0\frac{\alpha_s}{4\pi} - \beta_1\frac{\alpha_s^2}{(4\pi)^2} + \ldots$ mit $\left.\begin{aligned}\beta_0 &= 11 - \tfrac{2}{3}N_F\\ \beta_1 &= 102 - \tfrac{38}{3}N_F\end{aligned}\right\}$ hängen nicht von der Reg. ab!

$\vdots$

(höhere Ordnungen $\beta_2, \beta_3, \ldots$ hängen von Art der Regularisierung ab)

EFFEKTIVE KOPPLUNG

• niedrigste Ordnung:

$$g_s^2 = g_s^2(\mu^2)$$

$$t = -\int_{g_s}^{\bar{g}_s} \frac{dg'}{b\, g'^3} = \frac{1}{2b}\left[\frac{1}{\bar{g}_s^2} - \frac{1}{g_s^2}\right] \Rightarrow \bar{g}_s^2(g_s,t) = \frac{g_s^2}{1+\left(11-\frac{2}{3}N_f\right)\frac{g_s^2}{8\pi^2}\, t}$$

$$t = \frac{1}{2}\log\frac{Q^2}{\mu^2}$$

• höhere Ordnungen:

$$g_s^2(Q^2) = \frac{(4\pi)^2}{\beta_0 \log\frac{Q^2}{\Lambda^2}}\left[1 - \frac{\beta_1}{\beta_0^2}\,\frac{\log\log\frac{Q^2}{\Lambda^2}}{\log\frac{Q^2}{\Lambda^2}} + \dots\right]$$

mit $\Lambda^2 = \mu^2 \exp\left\{-\frac{16\pi^2}{g_s^2} + \frac{\beta_1}{\beta_0^2}\log(\beta_0 g_s^2)\right\}$

Q^2-Variation von Green's-Funktionen:

$$\left.\begin{aligned} \gamma_G(g_s^2) &= -d\, g_s^2 + \dots \\ d &= \left(\frac{13}{2} - \frac{2}{3}N_f\right)\frac{1}{(4\pi)^2} \\ \bar{g}_s^2(t) &= \frac{g_s^2}{1+2b g_s^2 t} \\ b &= \left(11 - \frac{2}{3}N_f\right)\frac{1}{(4\pi)^2} \end{aligned}\right\} \quad \begin{aligned} \int_0^t dt'\, \gamma_G(\bar{g}_s(g_s,t')) &= \int_0^t dt' (-d)\frac{g_s^2}{1+2b g_s^2 t'} \\ &= -\frac{d}{2b}\log(1+2b g_s^2 t) = -\log(1+2b g_s^2 t)^{\frac{d}{2b}} \end{aligned}$$

$$\Rightarrow \Gamma_R \propto e^{Dt}\, e^{\log(1+2b g_s^2 t)^{\frac{d}{2b}}} \xrightarrow[(t\to\infty)]{} e^{Dt}\, t^{\frac{d}{2b}}$$

$$\boxed{\Gamma_R \propto Q^D (\log Q)^{\frac{d}{2b}}}$$

Green's-Funktionen ändern sich in asymptotisch freien Theorien logarithmisch mit Q^2. [$\Leftarrow$ Fixpkt.-Theorie: $g = g^* \neq 0 \Rightarrow \Gamma_R \propto Q^D Q^{c^*}$]

B.) QCD bei kurzen Abständen

§1. Strukturfunktionen des Nukleons

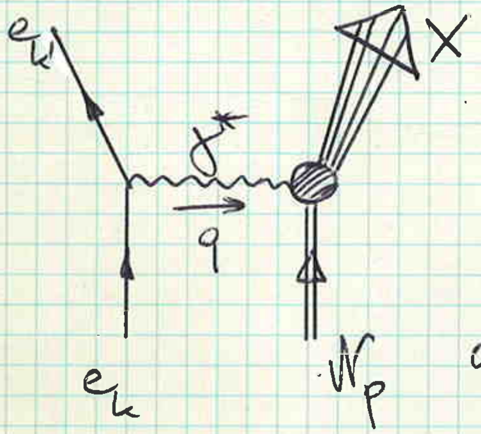
Asymptotische Freiheit:

(i) α_s klein $\Longrightarrow$ 0^{te} Approximation: approximativ freie Teilchen bei kleinen Abständen / hohen Energien

$\Longrightarrow$ PARTONMODELL

(ii) $\log Q^2$ – Abhängigkeit durch höhere Ordnungen

[o.B.d.A.: elektromagnetische Strukturfunktionen]



$$\mathcal{M}(X) = i e^2\, \bar{u}' \gamma^\mu u \,\frac{1}{q^2} \langle X | j_\mu | N_p \rangle$$

Wirkungsquerschnitt [E = e lab.-Energie]

$$d\sigma(e') = \frac{1}{4\pi E} \frac{d^3k'}{(2\pi)^3 2E'} \frac{1}{4} \sum_X (2\pi)^4 \delta_4(p+q-p_X) |\mathcal{M}_X|^2$$

$q = k - k' \qquad q^2 = -Q^2 < 0$

$$\frac{1}{4} \sum_X (2\pi)^4 \delta_4(p+q-p_X) |\mathcal{M}_X|^2$$

$$= \left(\frac{e^2}{Q^2}\right)^2 \underbrace{\frac{1}{4} \sum_{\text{spins}} [\bar{u}' \gamma^\nu u][\bar{u} \gamma^\mu u']}_{= \mathcal{L}_{\mu\nu} \text{ Leptontensor}} \underbrace{\sum_X \langle N | j_\mu | X \rangle \langle X | j_\nu | N \rangle (2\pi)^4 \delta_4(p+q-}_{= 8\pi W_{\mu\nu} \text{ Hadrontensor}}$$

Leptontensor: $\mathcal{L}_{\mu\nu} = k_\mu k'_\nu + k_\nu k'_\mu - (kk') g_{\mu\nu} \Leftarrow$ sym μ, ν; k, k'

Hadrontensor:

$$W_{\mu\nu} = \frac{1}{8\pi} \sum_{\text{spins}} \sum_X (2\pi)^4 \delta_4(p+q-p_X) \langle N_p | j_\mu^{elm} | X \rangle \langle X | j_\nu^{elm} | N_p \rangle$$

$$= \frac{1}{8\pi} \sum_{\text{spins}} \int d^4x\, e^{-iqx} \langle N_p | [j_\mu^{elm}(0), j_\nu^{elm}(x)] | N_p \rangle$$

Eigenschaften von $W_{\mu\nu}$:

(i) sym. Tensor in $p_\mu, q_\mu, g_{\mu\nu}$

(ii) Stromerhaltung: $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0 \qquad [\partial^\mu j_\mu^{elm} = 0]$

(iii) Tensor reell ($\Leftarrow$ Hermitezität des elm. Stroms)

Invariantenzerlegung:

allgem. Basis:

$$\underbrace{g_{\mu\nu} \quad q_\mu q_\nu}_{-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}} \qquad \underbrace{p_\mu p_\nu \quad p_\mu q_\nu + p_\nu q_\mu \quad q_\mu q_\nu}_{\left[p_\mu - q_\mu \frac{pq}{q^2}\right]\left[p_\nu - q_\nu \frac{pq}{q^2}\right]}$$

$$\boxed{W_{\mu\nu} = W_1\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right] + W_2\left[p_\mu - q_\mu\frac{pq}{q^2}\right]\left[p_\nu - q_\nu \frac{pq}{q^2}\right]}$$

W_i = Lorentz-skalare Strukturfunktionen

Variable: (i) Elektronzustand durch Energie und Streuwinkel charakterisiert

(ii) invariant: $Q^2 = -q^2 = +4EE' \sin^2\frac{\vartheta}{2}$ Streuwinkel

$\nu = pq = M(E-E')$ Energieverlust im e Sektor

Bereich: $\left.\begin{array}{l} Q^2 \geq 0 \\ \nu \geq 0 \end{array}\right\}$ $(p+q)^2 = W^2 \geq M^2$ (mind. N im Endzustand)

$M^2 + 2pq + q^2 \geq M^2 \implies 2\nu \geq Q^2$

$=$ elastisch

(iii) Skalenvariable:

$$\boxed{\begin{array}{lll} \text{Bjorken-Variable} & x = \frac{Q^2}{2\nu} & 0 \leq x \leq 1 \\ \text{relat. Energieverlust} & y = \frac{pq}{pk} & 0 \leq y < 1 \end{array}}$$

$$\boxed{\begin{array}{ll} \text{Strukturfkten:} & F_1(x, Q^2) = W_1(\nu, Q^2) \\ & F_2(x, Q^2) = \nu W_2(\nu, Q^2) \end{array}}$$

Wirkungsquerschnitt im Hochenergiebereich:

$$\boxed{\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2}{Q^4} s_{eN} \left[(1-y) F_2(x,Q^2) + y^2 x F_1(x,Q^2) \right]}$$

Interpretation der Strukturfkten:

Essenz von $eN \to e' + \text{alles}$ ist $\underline{\gamma^* + N \to \text{alles}}$

totaler Absorptionswq virtueller Photonen

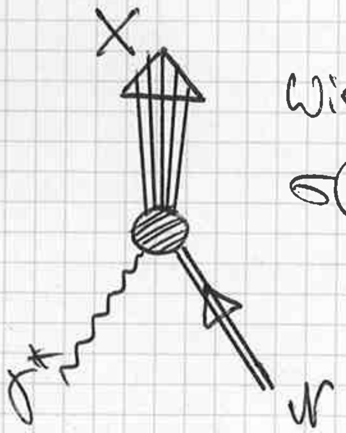
Wellenfunktionen virtueller raumartiger Photonen:

$$q_\mu = \left(\frac{\nu}{M}; 0, 0, \sqrt{Q^2 + \frac{\nu^2}{M^2}} \right) \quad \text{im Laborsystem}$$

$\longrightarrow$ transv. Polarisationen: $\varepsilon_\mu(\pm) = \frac{1}{\sqrt{2}}(0; 1, \pm i, 0)$

longitudinale Polarisation: $\varepsilon_\mu(L) = \frac{1}{\sqrt{Q^2}} \left(\sqrt{Q^2 + \frac{\nu^2}{M^2}}; 0, 0, \frac{\nu}{M} \right)$

Normierung: $\varepsilon_i \varepsilon_j^* = \pm\delta_{ij} \qquad \varepsilon_i q = 0 \qquad \varepsilon_\pm^* \varepsilon_\pm = -1 \qquad \varepsilon_L^2 = +1$



Wirkungsquerschnitt $\gamma^* + N \to \text{alles}$

$$\sigma(\gamma^* N) \propto \sum_X \varepsilon^{*\mu} \langle N | j_\mu | X \rangle \langle X | j_\nu | N \rangle \varepsilon_\nu (2\pi)^4 \delta_4(p + q - p_X)$$

$$\propto \varepsilon^{*\mu} W_{\mu\nu} \varepsilon_\nu$$

transv. Wq: $\sigma_\pm = \varepsilon_\pm^{*\mu} W_{\mu\nu} \varepsilon_\pm^\nu = W_1 = F_1 \geq 0 \quad [P_{elm}: \sigma_+ = \sigma_- = \frac{1}{2}\sigma_T]$

longit. Wq: $\sigma_L = \varepsilon_L^{*\mu} W_{\mu\nu} \varepsilon_L^\nu = -W_1 + \left(\frac{\nu^2}{Q^2} + M^2 \right) W_2 \geq 0$

$$\xrightarrow{(Q^2 \gg M^2)} -F_1 + \frac{1}{2x} F_2$$

R-Verhältnis:

$$\boxed{R = \frac{\sigma_L}{\sigma_T} \qquad R = \left(\frac{\nu^2}{Q^2} + M^2 \right) \frac{W_2}{W_1} - 1 \longrightarrow \frac{F_2 - 2x F_1}{2x F_1}}$$

Experimentelle Resultate:

1.) Bjorken Scaling:

Bjorken-Limes: $\left.\begin{array}{l} Q^2 \text{ groß} \\ x \text{ fest} \end{array}\right\}$

$$\boxed{\begin{aligned} \nu W_2(\nu, Q^2) &= F_2(x, Q^2) \rightsquigarrow F_2(x) \\ W_1(\nu, Q^2) &= F_1(x, Q^2) \underset{Bj}{\rightsquigarrow} F_1(x) \end{aligned}}$$

Scaling am deutlichsten ausgeprägt für $x \sim 0.25$

$\begin{cases} x \lesssim 0.25: F_2(x, Q^2) \text{ geringfügig mit } Q^2 \text{ ansteigend} \\ x \gtrsim 0.25: F_2(x, Q^2) \text{ geringfügig mit } Q^2 \text{ abfallend} \end{cases}$

Geringfügige, log. Brechung der Skaleninvarianz von QCD vorhergesagt.

2.) R-Verhältnis: $R(x, Q^2) \underset{Bj}{=} \frac{F_2(x) - 2x F_1(x)}{2x F_1(x)}$

Für große Q^2 strebt $R \to 0$, d.h. longit. γ^* Absorptionswq. verschwindet:

$$\boxed{\text{Callan-Gross Relation}: F_2 = 2x F_1}$$

3.) Neutron / Proton Verhältnis:

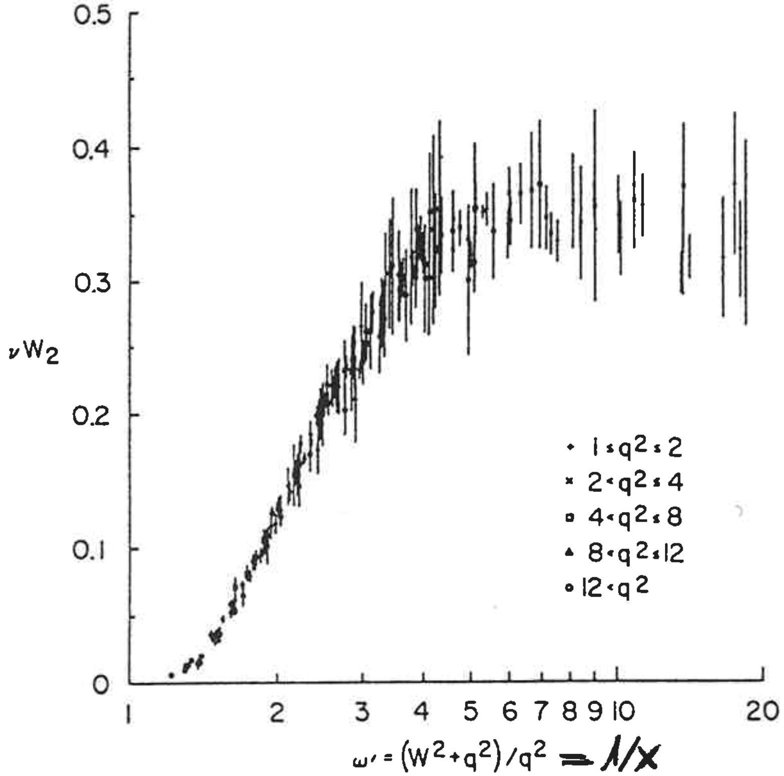
$F_2^N(x) / F_2^P(x)$ fällt vom Wert 1 bei $x = 0$ ab zum Wert $\gtrsim \frac{1}{4}$ für $x = 1$.

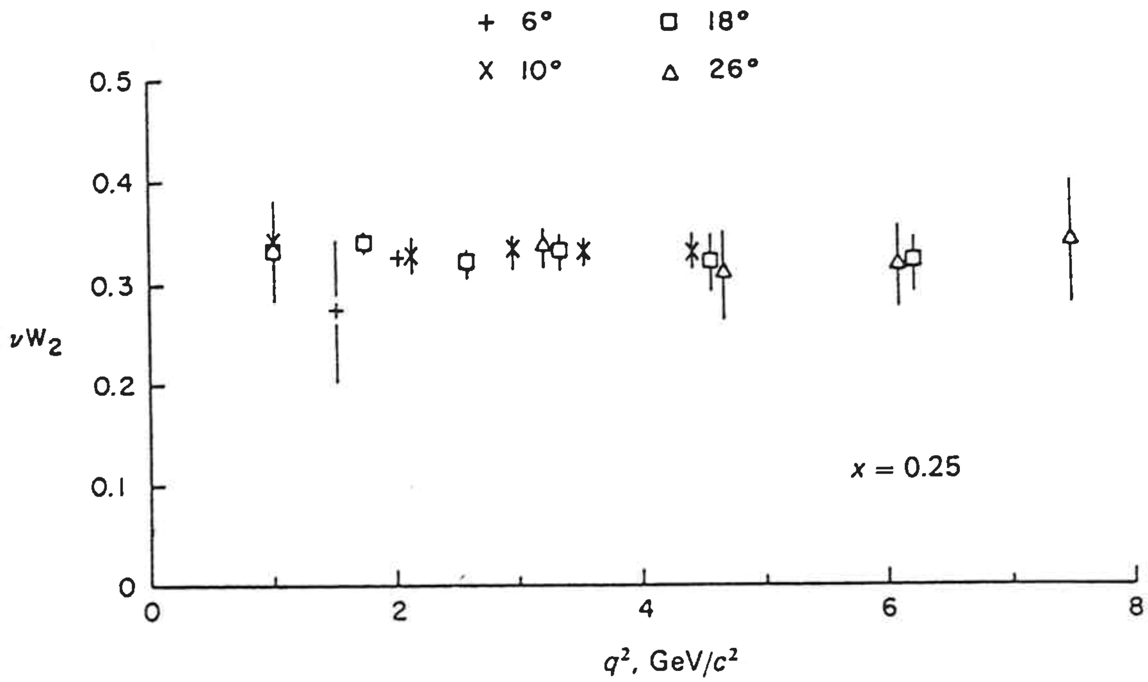
Klassisches Quark-Parton-Modell

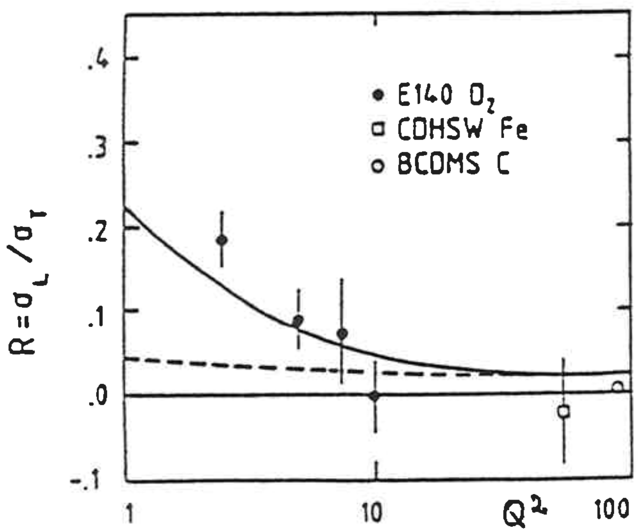
Basis:

e + pktf. → e + pktf.	eN → eN	eN → e + alles
$\frac{d\sigma^{pt}}{dQ^2} \sim \frac{1}{Q^4}$	$\frac{d\sigma^{el}}{dQ^2} \sim \frac{1}{Q^4} \lvert F(Q^2) \rvert^2 \sim \frac{d\sigma^{pt}}{dQ^2} \left(\frac{M^4}{Q^4}\right)^2$	$\frac{d\sigma}{dQ^2} \sim \frac{1}{Q^4} F_2(x) \sim \frac{d\sigma^{pt}}{dQ^2}$

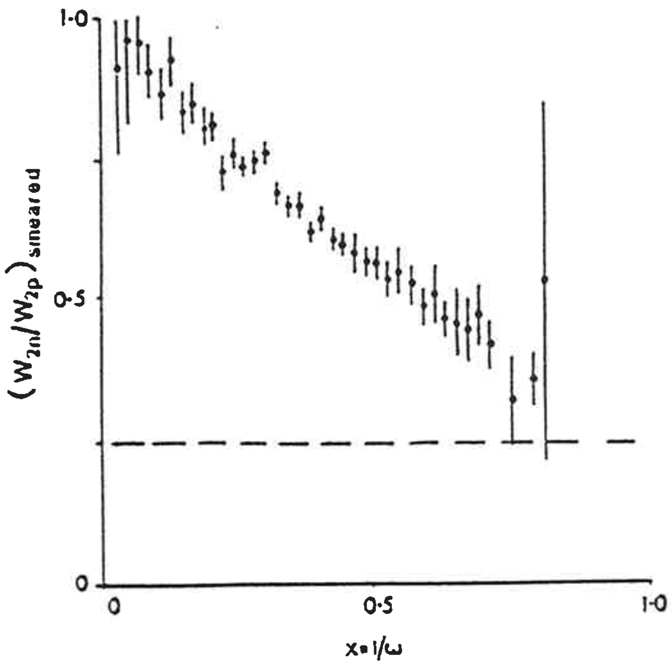
Scaling $F_2(x, Q^2) \approx F_2(x) \longrightarrow$ inklusiver Wq verhält sich bei $Q^2 \to \infty$ analog zum pktförmigen Wq [Q^2-Abfall um 8 Potenzen langsamer als elast. Nukleon-Wq]







Neutron / Proton Verhältnis:



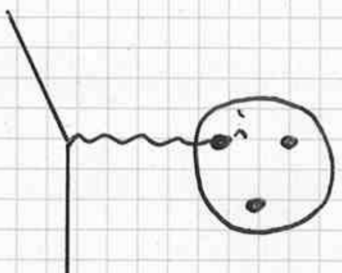
§2. Partonmodell der tiefinelastischen Lepton-Nukleon-Streuung

Im Quarkbild wird bei hohem Auflösungsvermögen die Reaktion durch Überlagerung der Streureaktionen an Quark-Konstituenten gebildet: Formulierung im <u>Partonmodell</u>.

ANMERKUNGEN

(i) Bei hohen Q^2 ist die Überlagerung <u>inkohärent</u>: $d\sigma = \sum_q d\sigma_q$

klassisch:



$$f = \sum_i f_i \qquad f_i = \int \frac{d^3\vec{r}}{2\pi} e^{-i\vec{q}\vec{r}} V_e(\vec{r}-\vec{r}_i)$$

$$= e^{-i\vec{q}\vec{r}_i} \int \frac{d^3\vec{r}\,'}{2\pi} e^{i\vec{q}\vec{r}\,'} V_e(\vec{r}\,')$$

$$= e^{-i\vec{q}\vec{r}_i} f_e$$

$$d\sigma = d\sigma_R \left| \sum_i e^{-i\vec{q}\vec{r}_i} \right|^2 \qquad f = F f_e$$

(a) $|\vec{q}|^{-1} \gg |\vec{r}_i|$: $d\sigma = N^2 d\sigma_R$ <u>kohärente</u> Überlagerung der Elementarprozesse bei kleinem Q^2

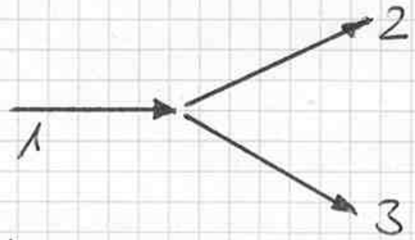
(b) $|\vec{q}|^{-1} \ll |\vec{r}_i|$: $\sum_{ij} e^{-i\vec{q}(\vec{r}_i-\vec{r}_j)} = \sum_{i=j} 1 + \sum_{i\neq j} e^{-i\vec{q}(\vec{r}_i-\vec{r}_j)}$

Beiträge interferieren sich zu Null

$$d\sigma = N\, d\sigma_R$$ <u>inkohärente</u> Überlagerung der Elementarprozesse bei großem Q^2

(ii) Wahrscheinlichkeitsbild:

"Zerlegung" eines Teilchens 1 in zwei Bestandteile 2 und 3 für großes P



$$p_1 = \left(P + \frac{m_1^2}{2P};\ 0_\perp, P\right)$$
$$p_2 = \left(|x|P + \frac{m_2^2+k_\perp^2}{2|x|P};\ k_\perp, xP\right)$$
$$p_3 = \left(|1-x|P + \frac{m_3^2+k_\perp^2}{2|1-x|P};\ -k_\perp, (1-x)P\right)$$

3-Impuls-Erhaltung

Energie-Sprung

Lösung der Born'schen Reihe <u>vor</u> Einführung der Zeitordnung

$$S_{fi} = \lim_{t\to+\infty} \langle f|U(t,-\infty)|i\rangle = \lim_{t\to+\infty} U(t,-\infty)_{fi}$$

$$U(t,-\infty)_{fi} = \delta_{fi} + (-i)\int_{-\infty}^{t} dt_1\, V_{fi}(t_1) + (-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \sum_n V_{fn}(t_1)V_{ni}(t_2) + \dots$$

$$= \delta_{fi} + (-i)\int_{-\infty}^{t} dt_1\, e^{i(E_f-E_i)t_1} V_{fi}$$
$$+ (-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \sum_n e^{i(E_f-E_n)t_1 + i(E_n-E_i)t_2} V_{fn}V_{ni} + \dots$$

$$= \delta_{fi} + e^{i(E_f-E_i)t}\left\{\frac{V_{fi}}{E_i-E_f+i\varepsilon} + \sum_n \frac{V_{fn}}{E_i-E_f+i\varepsilon}\,\frac{V_{ni}}{E_i-E_n+i\varepsilon} + \dots\right\}$$

$$\lim_{t\to\infty} \frac{e^{i(E_f-E_i)t}}{E_i-E_f+i\varepsilon} = \frac{1}{i}\lim_{t\to\infty}\int_{-\infty}^{t} dt_1\, e^{i(E_f-E_i)t_1} = -2\pi i\,\delta(E_f-E_i)$$

$$S_{fi} = \delta_{fi} - 2\pi i\,\delta(E_f-E_i)\left\{V_{fi} + \sum_n \frac{V_{fn}V_{ni}}{E_i-E_n+i\varepsilon} + \dots\right\}$$

"old-fashioned perturbation theory"

$$\Delta E = E_1 - (E_2+E_3) = P(1-|x|-|1-x|) \quad \text{für } x<0 \text{ und } x>1:\ \Delta E \sim P$$
$$= \frac{1}{2P}\left[m_1^2 - \frac{m_2^2+k_\perp^2}{x} - \frac{m_3^2+k_\perp^2}{1-x}\right] \quad \text{für } 0<x<1:\ \Delta E \sim P^{-1}$$

führend

$\longrightarrow$ Lebensdauer $\tau_L \sim \frac{1}{\Delta E} \sim \frac{P}{\langle k_\perp^2\rangle}$ sehr lang im eP-c.m.s.

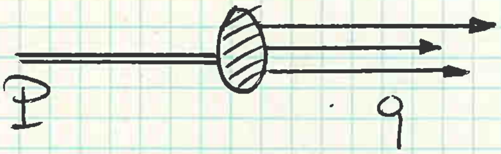
[$x<0$, $x>1$: eins der Tochterteilchen läuft rückwärts]

$\Rightarrow$ Bei schnell bewegten Teilchen dominiert diejenige Zerlegung, bei der die Tochterteilchen, parallel zum Mutterteilchen, einen Energie/Impuls-Bruchteil x mit $0<x<1$ aufnehmen.

<u>Quark-Partonmodell</u>:

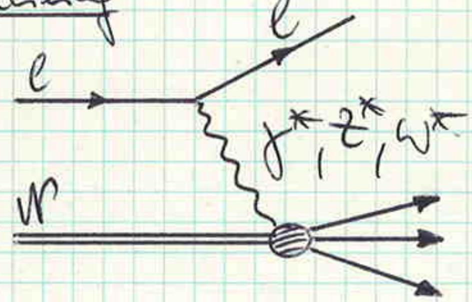
(i) Im schnell bewegten Koordinatensystemen kann Nukleon in wechselwirkungsfreie parallele Partonen zerlegt werden, an denen Leptonen inkohärent streuen.

(ii) Partonen können mit punktförmigen Quarks identifiziert werden.



Feynman
Bjorken, Paschos

<u>Tiefinelastische Lepton-Nukleon-Streuung</u>



Lorentz-Invariant
punktförmiger Leptonstrom
Spin 1 Austausch $\Rightarrow$

$$\frac{d\sigma^{elm}}{dx\,dy} = \frac{4\pi\alpha^2}{Q^4}\, s \left\{ (1-y) F_2^{elm}(x,Q^2) + y^2 x F_1^{elm}(x,Q^2) \right\}$$

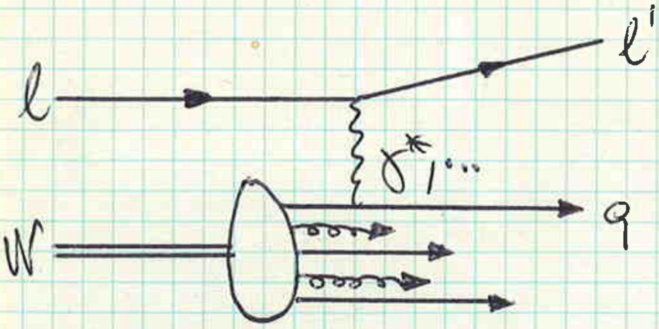
$$\frac{d\sigma_{cc}^{\nu/\bar\nu}}{dx\,dy} = \frac{G_F^2 s}{2\pi} \left\{ (1-y) F_2^{\nu/\bar\nu}(x,Q^2) + y^2 x F_1^{\nu/\bar\nu}(x,Q^2) \pm \frac{1-(1-y)^2}{2} x F_3^{\nu/\bar\nu}(x,Q^2) \right\}$$

$F_i = F_i(x,Q^2)$ elm. und schwache Strukturfunktionen

transversal: $F_T = F_1$

longitudinal: $F_L = F_2 - 2xF_1$ $\qquad R = \frac{F_L}{2xF_T}$

Quark-Parton Bild:



WW-Zeit:

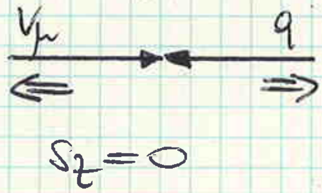
$$\left.\begin{aligned} \nu &= pq = P(q^0+q^3) = \frac{Q^2}{2x} \\ Q^2 &= -(q^0)^2 + (q^3)^2 \end{aligned}\right\}$$

$$\Delta E^* = -q^0 = xP - \frac{Q^2}{4xP} \sim xP \quad (x>0)$$

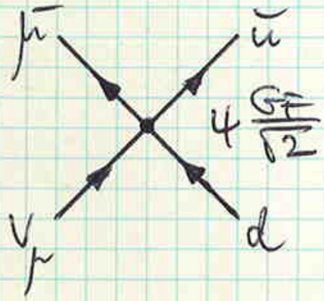
$$\left.\begin{aligned} \tau_{WW} &\sim \frac{1}{xP} \\ \tau_L &\sim \frac{P}{\langle k_\perp^2 \rangle} \end{aligned}\right\} \quad \boxed{\tau_{WW} \ll \tau_L} \longrightarrow$$ Quarks reell bei kurzen Abstandsprozessen

Quark-Parton Wirkungsquerschnitte

$\boxed{\nu_\mu d \to \mu^- u}$



$$\frac{d\sigma^q}{d\cos\vartheta_*} \sim \frac{1}{2s_*}\left(\frac{4G_F}{\sqrt{2}}\right)^2 \frac{s_*^2}{8\pi} \sim \frac{G_F^2 s_*}{\pi}$$



$$y = \frac{pq}{pk}\Big|_{IMF} = \frac{p_q q}{p_q k}\Big|_{CM} = \frac{\varepsilon_*^2(1-\cos\vartheta_*)}{2\varepsilon_*^2}$$

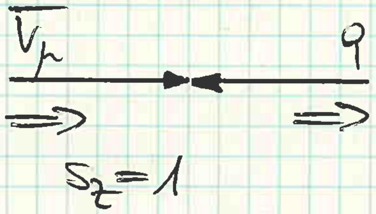
$$y = \frac{1}{2}(1-\cos\vartheta_*)$$

$$\frac{d\sigma^q}{dy} = \frac{G_F^2 s_*}{\pi}$$

$\nu_\mu d \to \mu^- u$ LL

$\bar\nu_\mu \bar d \to \mu^+ \bar u$ RR

$\boxed{\bar\nu_\mu u \to \mu^+ d}$



$$\frac{d\sigma^q}{d\cos\vartheta_*} \sim (1+\cos\vartheta_*)^2 \sim (1-y)^2$$

keine Rückwärtsstreuung: RL, LR (νq)

$$\frac{d\sigma^q}{dy} = \frac{G_F^2 s_*}{\pi}(1-y)^2$$

$\boxed{eq \to eq}$ beide Helizitäten inkohärent

$$\frac{d\sigma^q}{dy} = \frac{2\pi\alpha^2 e_q^2}{Q^4} s_* \left[1+(1-y)^2\right]$$

Zusammensetzung:

iMF: CM(ℓ, N) $f_q(\xi)d\xi$ = Anzahl der Quarks q im

Breit-System: $q=(0,0,0,q)$ Impulsintervall $d\xi$ um ξ: $p_q=\xi P$

⋮

$$\frac{d\sigma}{dx\,dy}=\sum_q\int_0^1 d\xi\, f_q(\xi)\,\frac{d\sigma^q(s_*=\xi s)}{dy}\,\delta_1\!\left(x-\frac{Q^2}{2\nu}\right)$$

$$\frac{Q^2}{2\nu}=\xi\frac{Q^2}{2\nu_q}=\xi$$

↳ Elastizitätsbedingung: $(p_q+q)^2=p_q^2$

$$-Q^2+2qp_q=0 \implies \frac{Q^2}{2\nu_q}=1$$

$$\boxed{\frac{d\sigma}{dx\,dy}=\sum_q f_q(x)\,\frac{d\sigma^q(s_*=xs)}{dy}}$$

Bjorken-Variable bestimmt den Relativimpuls des gestreuten Quarks: $\xi=x$

ν: $$\frac{d\sigma}{dx\,dy}=\frac{G_F^2 s}{4}\left\{x d(x)+(1-y)^2 x\bar{u}(x)\right\}$$

$\bar{\nu}$: $$\frac{d\sigma}{dx\,dy}=\frac{G_F^2 s}{4}\left\{(1-y)^2 x u(x)+x\bar{d}(x)\right\}$$

e: $$\frac{d\sigma}{dx\,dy}=\frac{2\pi\alpha^2 s}{Q^4}\sum_q e_q^2\, x f_q(x)\left[1+(1-y)^2\right]$$

Auswertung:

$F_1(x,Q^2)$ unabhängig von Q^2: Skaleninvariant

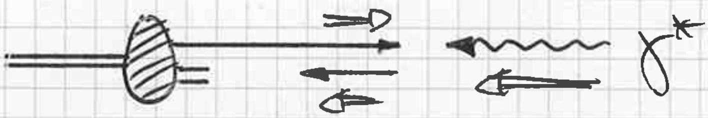
$F_2=2xF_1$ Callan-Gross Relation

$F_2^{elm}=\sum_q e_q^2\, x f_q(x)$ $F_2^{\nu}=2x(d+\bar{u})$ $xF_3^{\nu}=+2x(d-\bar{u})$

$F_2^{\bar{\nu}}=2x(u+\bar{d})$ $xF_3^{\bar{\nu}}=-2x(u-\bar{d})$

Physikalische Interpretation:

1.) Callan-Gross Relation mißt $\underline{\underline{\text{Quarkspin} = \frac{1}{2}}}$



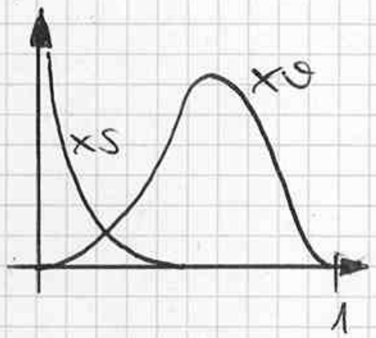
Breitsystem $q = (0;0,0,q)$: Strom $\bar{q}\gamma_\mu q$ chiral erhaltene masselose, parallele Quarks

$\Delta S_z = 1 \Longrightarrow S_z(\gamma^*) = 1, \neq 0$

$\Longrightarrow \sigma_T \neq 0, \sigma_L = 0 \Longrightarrow R = \frac{\sigma_L}{\sigma_T} = 0$

[spinlose Partonen: $\sigma_T = 0, \sigma_L \neq 0 \rightarrow R = \infty$ ↯]

2.) Valenzquarks: $f = v + s$ $v(x)$ Valenzquarkverteilung



$\int_0^1 dx\, v_d = 1$

$\int_0^1 dx\, v_{\bar{u}} = 2$

$v(x) \sim x^{-1/2}(1-x)^3$

$s(x)$ divergente Seequarks aus Quantenfluktuat.

$\int_\varepsilon^1 dx\, s \sim \log \varepsilon$

$s(x) \sim x^{-1}(1-x)^{7\ldots 10}$ ≈ 0 für $x \gtrsim 0.2$

gedrittelte elektrische Quarkladung:

nukleares Target $F_2^{elm} \approx x\left[\frac{4}{9}\frac{\bar{u}+d}{2} + \frac{1}{9}\frac{d+\bar{u}}{2}\right] = \frac{5}{18}x(\bar{u}+d)$

Valenzbereich

$N = \frac{1}{2}(P+N)$ $F_2^\nu \approx 2x\frac{d+\bar{u}}{2} = x(\bar{u}+d)$

$$\boxed{F_2^{elm} \approx \frac{5}{18}F_2^\nu}$$

3.) <u>3 Quarks im Nukleon</u>

nukl. Target $\qquad \sigma_\nu \approx \frac{G_F^2 s}{4} \int_0^1 dx\, x\, \frac{u+d}{2} \qquad \sigma_{\bar\nu} \approx \frac{1}{3}\sigma_\nu$

Val-dominant $\qquad \sigma_{\bar\nu} \approx \frac{1}{3} \frac{G_F^2 s}{4} \int_0^1 dx\, x\, \frac{u+d}{2}$

Spin: Quarks keine Antiquarks

<u>Summenformeln:</u> ("exakt")

Baryon-Zahl $\qquad 1 = \int_0^1 dx\, \frac{1}{3}\left[(u-\bar u)+(d-\bar d)+(s-\bar s)\right]$

Isospin $\qquad \pm\frac{1}{2} = \int_0^1 dx \left[\frac{1}{2}(u-\bar u)-\frac{1}{2}(d-\bar d)\right]$

Strangeness $\qquad 0 = \int_0^1 dx\,(s-\bar s)$

Auflösung Proton: $\int_0^1 dx\,(u-\bar u)=2 \qquad \int_0^1 dx\,(d-\bar d)=1 \qquad \int_0^1 dx\,(s-\bar s)=0$

nukl. Target: $\int_0^1 dx\, F_3^\nu = \int_0^1 dx\left[d+u-\bar u-\bar d\right] = \int_0^1 dx\left[(u-\bar u)+(d-\bar d)\right]$

<u>Gross-Llewellyn-Smith:</u> $\int_0^1 dx\, F_3^\nu = 3$

4.) <u>Impuls-Summenregel:</u> $\qquad 1 = \sum_{q,\bar q} \int_0^1 d\xi\, \xi\, f_q(\xi) + \int_0^1 d\xi\, \xi\, f_g(\xi)$

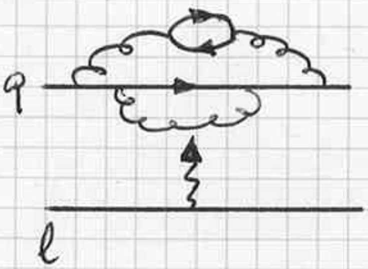
flavor-neutrale Materie: Bindungsenergie

Messung: $\int_0^1 d\xi\, \xi\, f_g(\xi) \approx \frac{1}{2}$

<u>50% der Nukleonenergie besteht im schnell bewegten Teilchen aus flavor-neutraler Bindungsenergie: GLUONEN</u>

§3. Skalenbrechung: Altarelli-Parisi-Gleichungen

Idee: Partonquarks sind im Nukleon von Gluonwolke umgeben;



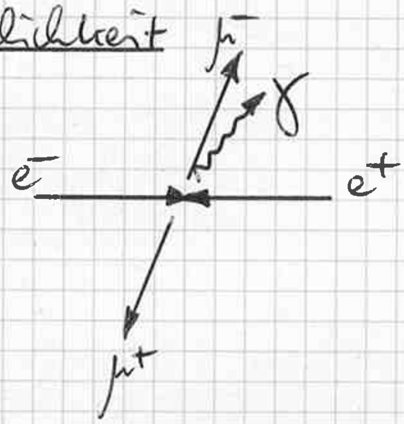
bei hinreichend kleinem Q^{-1} werden mehr und mehr der Quantenfluktuationen aufgelöst

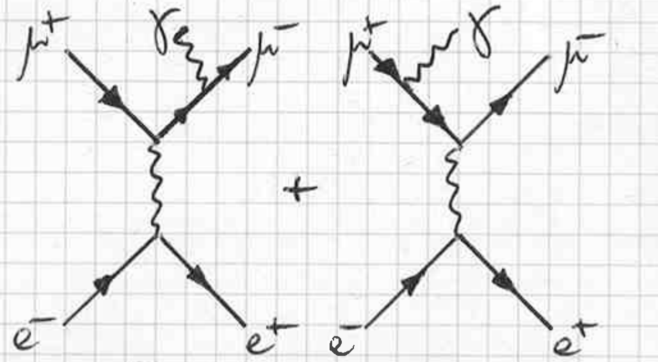
$\Rightarrow$ Impulsspektrum von Quarks, Gluonen ändert sich mit Q^{-1}: mikroskopische Partonverteilungen sind Q^2-abhängig.

Splitting-Wahrscheinlichkeit

$e^+e^- \to \mu^+ \underbrace{\mu^- \gamma}$

$\theta_{\mu\gamma}$ klein fest





$x_{1,2} = \frac{E_\pm}{E}$ $\qquad z = \frac{E_\gamma}{E}$

$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dx_1 dx_2} = \frac{\alpha}{2\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

$$x_\perp = \frac{2}{x_1}\sqrt{(1-x_1)(1-x_2)(1-z)} = \frac{p_\perp}{E}$$

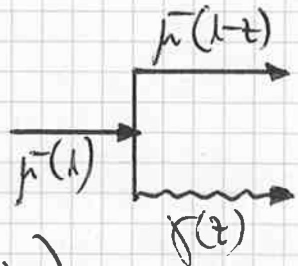
$$\log x_\perp^2 \approx \log(1-x_1)$$

$$d\log p_\perp^2 \approx \frac{dx_1}{1-x_1}$$

$$x_1+x_2+z = 2$$

Fragmentation: $x_2 \approx 1-z$

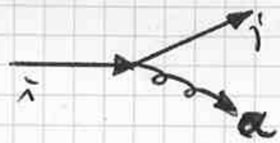
$$d\sigma = \sigma_0 \int^{Q^2} \frac{dp_\perp^2}{p_\perp^2}\,\frac{\alpha}{2\pi}\,\frac{1+(1-z)^2}{z}\,dz$$



Wq = μ-Paar-Wq $*$ Teilchenfluß ($\mu \to \mu\gamma$)

Teilchenflußzuwachs bei $Q^2 \to Q^2 + \delta Q^2$: $\frac{\delta N(\mu\to\mu\gamma)}{\delta \log Q^2} = \frac{\alpha}{2\pi}\frac{1+(1-z)^2}{z}dz$

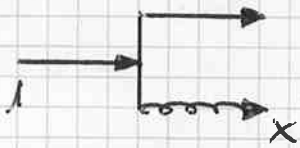
Quark fragmentation: Farbmittel/summe



$$\sum_{k,a} T^a_{ik} T^a_{kj} = \frac{4}{3}\delta_{ij}$$

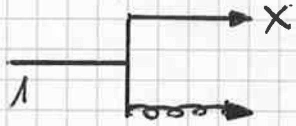
<u>QCD Splitting-Wahrscheinlichkeiten</u>: $\dfrac{\delta N}{\delta \log \frac{Q^2}{\Lambda^2}} = \dfrac{\alpha_s(Q^2)}{2\pi} P(x)\,dx$

$q \to q + g(x)$



$P_{gq} = \frac{4}{3}\,\frac{1+(1-x)^2}{x}$ Bremsstrahlung $x \to 0$

$q \to q(x) + g$

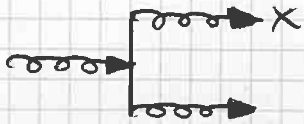


$P_{qq} = \frac{4}{3}\,\frac{1+x^2}{1-x}$ " $x \to 1$

$g \to q(x) + \bar{q}$

$P_{qg} = P_{\bar{q}g} = \frac{1}{2}\left[x^2+(1-x)^2\right]$ endlich

$g \to g(x) + g$



$P_{gg} = 6\,\frac{\left[1-x+x^2\right]^2}{x(1-x)}$ Bremsstrahlung $x \to 0,1$

<u>Altarelli-Parisi Mastergleichungen für Partondichten</u>:

$Q^2 \to Q^2 + \delta Q^2$

$$\frac{\partial q(x,Q^2)}{\partial \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\, \delta_1(x-yz)\left\{P_{qq}(y)\,q(z,Q^2) + P_{qg}(y)\,g(z,Q^2)\right\}$$

$$-\frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy'\, P_{qq}(y')\, q(x,Q^2)$$

$$\int_0^1 dy'\, P_{qq}(y')\, q(x,Q^2) = \int_0^1 dy \int_0^1 dz\, \delta_1(x-yz)\,\delta(y-1)\left[\int_0^1 dy'\, P_{qq}(y')\right] q(z,Q^2)$$

$$\frac{\partial q(x,Q^2)}{\partial \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\, \delta_1(x-yz)\left\{P^R_{qq}(y)\, q(z,Q^2) + P_{qg}(y)\, g(z,Q^2)\right\}$$

$$\frac{\partial g(x,Q^2)}{\partial \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\, \delta_1(x-yz)\left\{P_{gq}(y)\sum_{fl}\left[q(z,Q^2)+\bar q(z,Q^2)\right] + P^R_{gg}(y)\, g(z,Q^2)\right\}$$

$$P^R_{qq}(y) = P_{qq}(y) - \delta(y-1)\int_0^1 dy'\, P_{qq}(y')$$

$$P^R_{gg}(y) = P_{gg}(y) - \delta(y-1)\left[\frac{1}{2}\int_0^1 dy'\, P_{gg}(y') + N_F\int_0^1 dy'\, P_{qg}(y')\right]$$

$$\alpha_s(Q^2) = \frac{12\pi}{(33-2N_F)\log\frac{Q^2}{\Lambda^2}}$$

teilweise Entkopplung: $\delta = q - q'$ Non-Singulett

$\left.\begin{array}{l}\Sigma = \sum_{fl}(q+\bar q)\\ g\end{array}\right\}$ gekoppeltes Singulettsatz

<u>LÖSUNGEN</u>:

Übergang zu <u>Momenten</u>: $q(N,Q^2) = \int_0^1 dx\, x^{N-1} q(x,Q^2)$

wandelt Integro-Differentialgleichungssystem in System gewöhnlicher Differentialgleichungen um.

<u>Natürliche Variable</u>: $s = \log\frac{\log Q^2}{\log Q_0^2}$ [Q_0 = Referenz-Imp. Übertrag]

[bei fester Kopplungskonstante wäre $t = \log Q^2$ die natürl. Variable]

1.) Nonsingulett – Dichte:

$$\frac{\partial}{\partial s}\,\delta(N,Q^2) = \frac{6}{33-2N_F}\int_0^1 dy\, y^{N-1}\, P^R_{qq}(y)\;\delta(N,Q^2)$$

$$\hookrightarrow = \frac{6}{33-2N_F}\,\frac{4}{3}\left[-\frac{1}{2}+\frac{1}{N(N+1)}-2\sum_{j=2}^{N}\frac{1}{j}\right] \equiv -d_{NS}(N)$$

$$\frac{\partial}{\partial s}\,\delta(N,Q^2) = -d_{NS}(N)\,\delta(N,Q^2) \Longrightarrow \delta = \delta_0\, e^{-s d_{NS}}$$

$$\boxed{\begin{aligned}\delta(N,Q^2) &= \delta(N,Q_0^2)\left[\frac{\log Q^2}{\log Q_0^2}\right]^{-d_{NS}(N)}\\ &= \delta(N,Q_0^2)\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{d_{NS}(N)}\end{aligned}}$$

$\Leftarrow$ logarithm. Verletzung der Bjorken-Skalen=invarianz

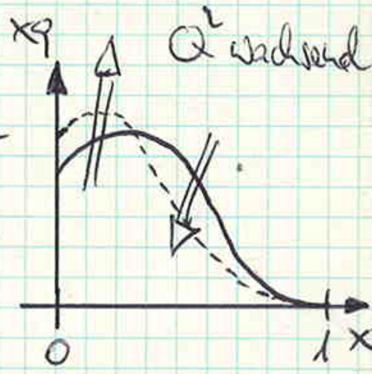
Interpretation:

(i) asymptotische Freiheit $\Longrightarrow \left[\frac{\log Q^2}{\log Q_0^2}\right]^{-d}$

fix. Kopplung $\Longrightarrow \left[\frac{Q^2}{Q_0^2}\right]^{-d}$

(ii) $d_{NS}(N=1) = 0$: Nettoquarkanzahl ungeändert

$d_{NS}(N>1) > 0$: Momente fallen mit wachsendem Q^2 ab



(iii) Momenten-Vergleich: Test der anomalen Dimensionen

Q^2-Abhängigkeit der Strukturfunktionen

F

2.) Quarksingulett und Gluon-Dichten:

$$\frac{\partial}{\partial s}\begin{pmatrix}\Sigma\\ G\end{pmatrix} = -\begin{pmatrix} d_{QQ} & d_{QG}\\ d_{GQ} & d_{GG}\end{pmatrix}\begin{pmatrix}\Sigma\\ G\end{pmatrix} \quad \text{mit } \Sigma=\Sigma(N,Q^2) \text{ etc.}$$

$$d_{QQ}(N) = -\frac{c}{33-2N_f}\int_0^1 dy\, y^{N-1} P^R_{qq}(y) = \frac{4}{33-2N_f}\left[1-\frac{2}{N(N+1)}+4\sum_{j=2}^{N}\frac{1}{j}\right] = d_{NS}(N)$$

$$d_{QG}(N) = \quad \sim \quad 2N_f P_{qg} \quad = -\frac{6N_f}{33-2N_f}\,\frac{N^2+N+2}{N(N+1)(N+2)}$$

$$d_{GQ}(N) = \quad \sim \quad P_{gq} \quad = -\frac{8}{33-2N_f}\,\frac{N^2+N+2}{(N-1)N(N+1)}$$

$$d_{GG}(N) = \quad \sim \quad P^R_{gg} \quad = \frac{9}{33-2N_f}\left\{\frac{1}{3}-\frac{4}{N(N-1)}-\frac{4}{(N+1)(N+2)} + 4\sum_{j=2}^{N}\frac{1}{j}+\frac{2N_f}{9}\right\}$$

Lösung des Systems mittels Exponentialansatz $\Longrightarrow$

$$\boxed{\begin{aligned}\Sigma &= \frac{1}{\mu_+-\mu_-}\left\{\left[-\mu_-\Sigma_0+G_0\right]e^{-d_+ s}+\left[\mu_+\Sigma_0-G_0\right]e^{-d_- s}\right\}\\ G &= \frac{1}{\mu_+-\mu_-}\left\{\mu_+\left[-\mu_-\Sigma_0+G_0\right]e^{-d_+ s}+\mu_-\left[\mu_+\Sigma_0-G_0\right]e^{-d_- s}\right\}\end{aligned}}$$

Eigenwerte: $d_\pm(N) = \frac{1}{2}\left[(d_{GG}+d_{QQ})\pm\sqrt{(d_{GG}-d_{QQ})^2+4d_{QG}d_{GQ}}\right]$

Eigenvektoren: $\mu_\pm(N) = \frac{d_\pm - d_{QQ}}{d_{QG}} = \frac{1}{2}\,\frac{d_{GG}-d_{QQ}\pm\sqrt{(d_{GG}-d_{QQ})^2+4d_{QG}d_{GQ}}}{d_{QG}}$

PHYSIKALISCHE FOLGERUNGEN:

(a) Impuls-Summenregel:

$$\left.\begin{aligned} d_-(2) &= 0 \\ \mu_+(2) &= -1 \end{aligned}\right\} \quad \underline{\underline{\Sigma(2) + G(2) = 1}} \qquad \text{folgt aus } \Sigma_0(2) + G_0(2) = 1$$

(b) Asymptotische Impuls-Verteilung:

$$\left.\begin{aligned} d_-(2) &= 0 \\ d_+(2) &> 0 \end{aligned}\right\} \quad \Sigma(2) \to \frac{1}{\mu_-(2) - \mu_+(2)} = \frac{3N_f}{16 + 3N_f} = \frac{3}{7} \text{ für } N_f = 4$$

$$\underline{\underline{G(2) \to \frac{\mu_-(2)}{\mu_-(2) - \mu_+(2)} = \frac{16}{16 + 3N_f} = \frac{4}{7} \text{ für } N_f = 4}}$$

(c) Messung des Gluon-Momente:

Tiefinelastische eN-Streuung: keine direkte Möglichkeit, Gluon-Verteilung zu messen.

indirekt:
- Impuls-Summenregel: $G(2, Q_0^2) = 1 - \Sigma(2, Q_0^2)$
- beeinflußt Stärke, mit der $\Sigma(N, Q^2)$ sich mit Q^2 ändert durch Aufbau von See-Verteilung.
- [$G \to Q\bar{Q}$ Spaltung in schwere Quarks]

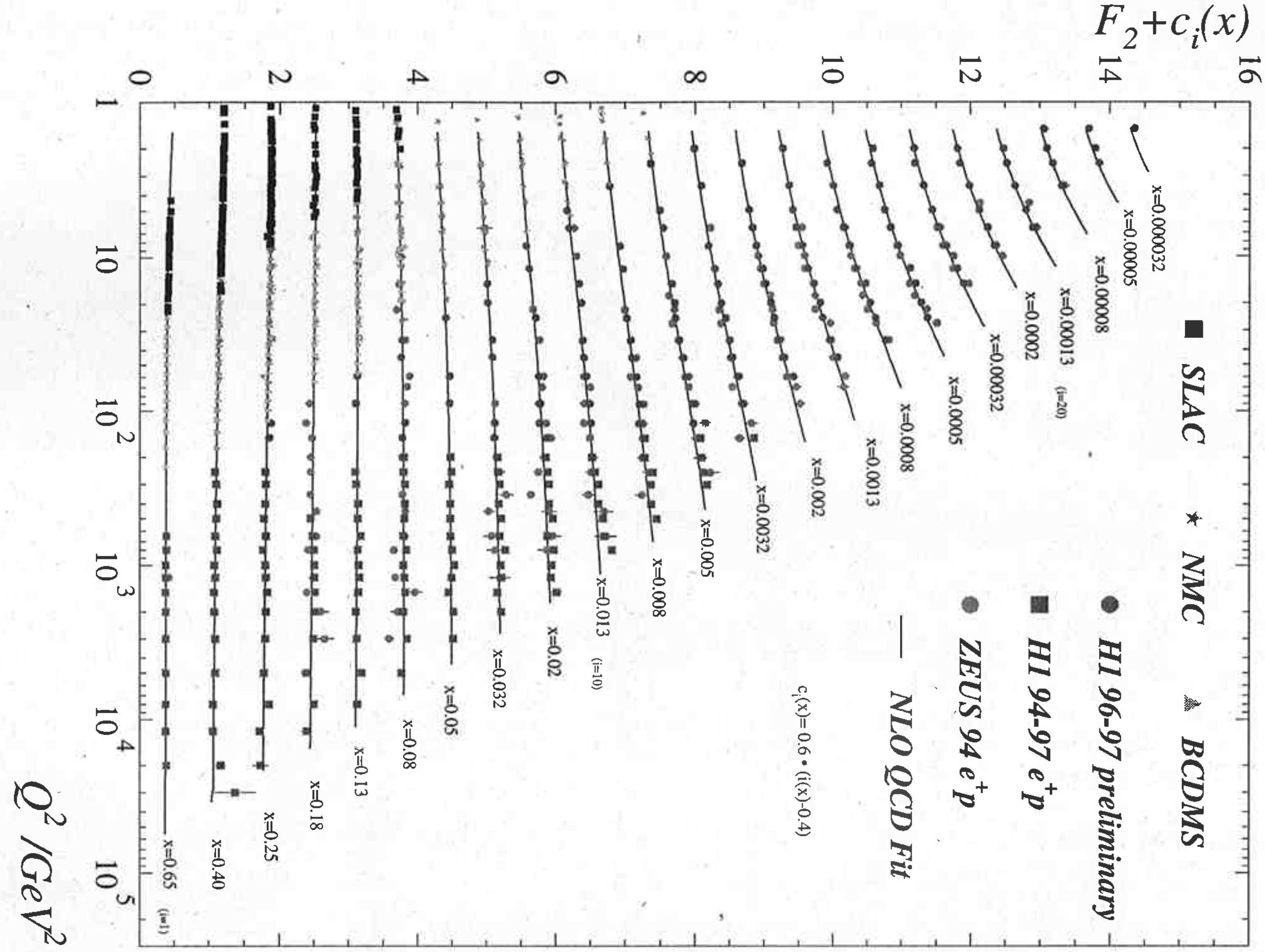
$$F_2^S(x, Q^2) = \frac{5}{18} x \left[u + \bar{u} + d + \bar{d}\right] \quad \text{o.B.d.A.} \quad = \frac{5}{18} x\, \Sigma(x, Q^2)$$

$$F_2^S(N, Q^2) = \frac{5}{18} \Sigma(N, Q^2)$$

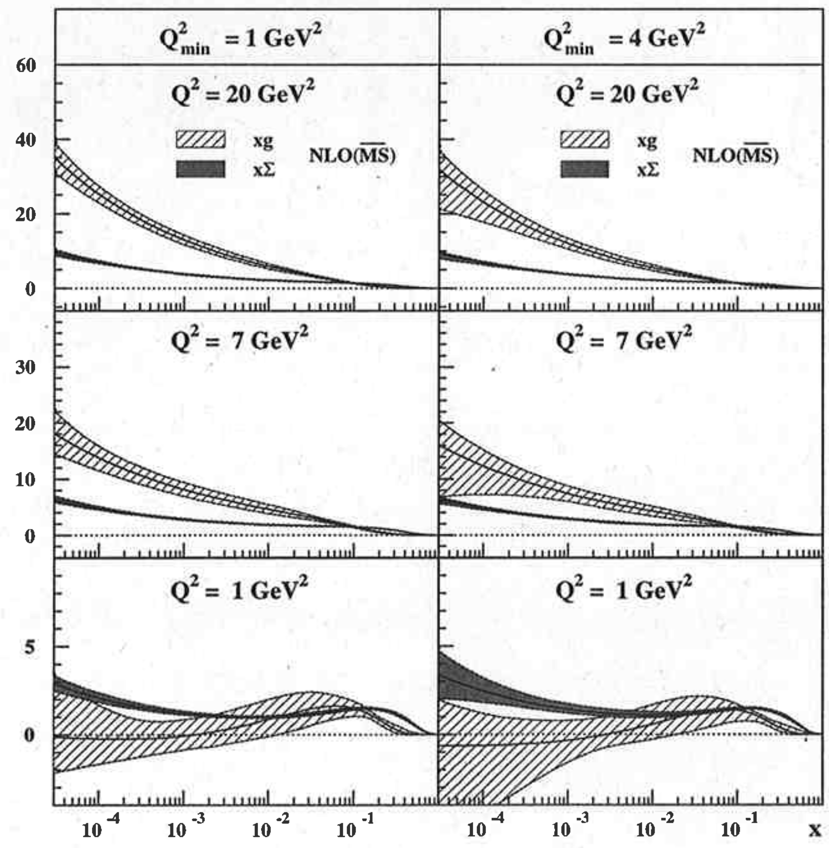
$$= \frac{5}{18} \left\{ \frac{-\mu_- e^{-d_+ s} + \mu_+ e^{-d_- s}}{\mu_+ - \mu_-} \underbrace{\Sigma(N, Q_0^2)}_{= \frac{18}{5} F_2^S(N, Q_0^2)} + \frac{e^{-d_+ s} - e^{-d_- s}}{\mu_+ - \mu_-} G(N, Q_0^2) \right\}$$

$$\boxed{\frac{1}{A_N(s)} F_2^S(N, Q^2) = F_2^S(N, Q_0^2) + \frac{B_N(s)}{A_N(s)} G(N, Q_0^2)}$$

Linke Seite gemessen als Gerade in $B_N(s)/A_N(s)$ mit G-Verteilung als Steigung


$F_2+c_i(x)$
SLAC
NMC
BCDMS
H1 96-97 preliminary
H1 94-97 e^+p
ZEUS 94 e^+p
NLO QCD Fit
$c_i(x) = 0.6 \cdot (i(x)-0.4)$
Q^2 /GeV^2

ZEUS 1995


Q^2_{min} = 1 GeV^2
Q^2_{min} = 4 GeV^2
Q^2 = 20 GeV^2
xg
xΣ
NLO($\overline{MS}$)
Q^2 = 7 GeV^2
Q^2 = 1 GeV^2
x

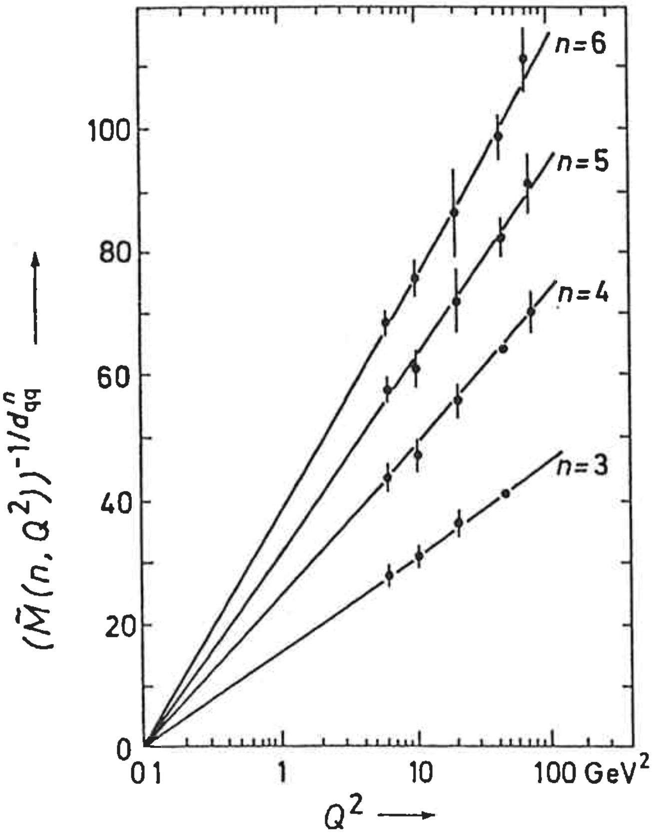


Bild 19-7
Die Momente der Strukturfunktion $F_3^{(\nu N)}$, gemessen in Neutrino-Eisen-Streuung (nach de Groot 1979)

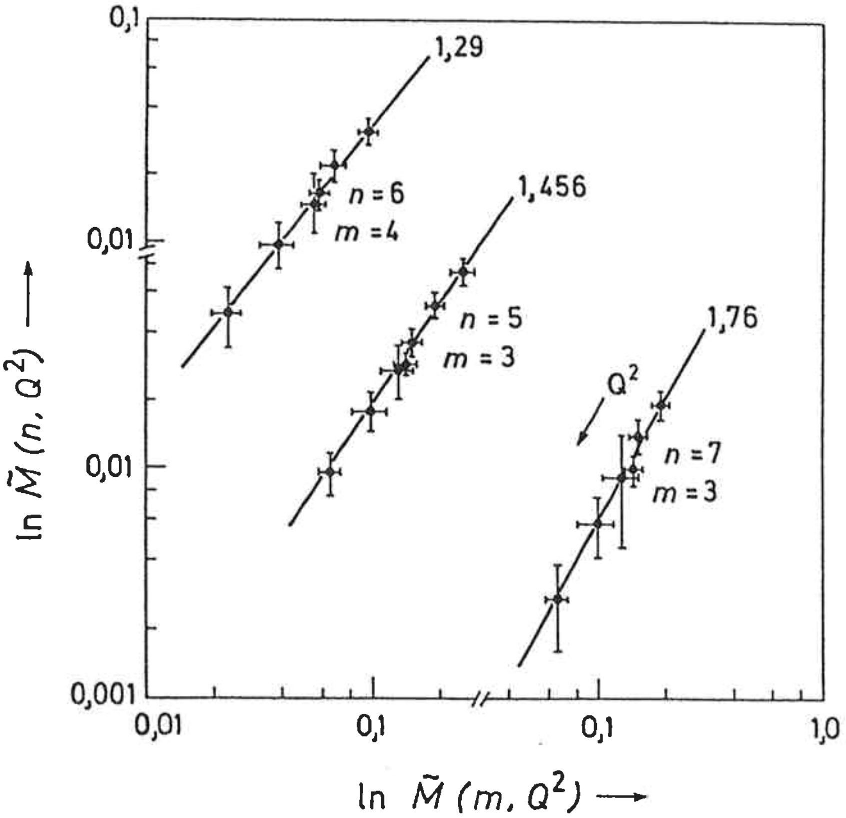


Bild 19-8
Logarithmen von Momenten der Strukturfunktion F_3 gegeneinander aufgetragen. Die QCD-Vorhersagen sind gerade Linien mit berechenbarem Anstieg, wie angegeben (nach Bosetti 1978).

§4. Faktorisierungstheoreme der QCD

QCD-Korrekturen zur tiefinelastischen eN-Streuung:

$$\boxed{\text{DIMENSIONALE REGULARISIERUNG}}$$

Idee: analytische Fortsetzung 4-dim. $\to$ n-dim. $[n=4-2\varepsilon]$

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^nk}{(2\pi)^n}$$

divergentes Integral: $\int \frac{d^4k}{k^4} \to \int \frac{d^nk}{k^4} \propto \frac{1}{n-4} = \frac{-1}{2\varepsilon}$

$\Longrightarrow$ UV-Singularitäten als Pole für $\varepsilon \to 0+$

[Eichinvarianz exakt erhalten]

Feynman-Parametrisierung

$$\sim \int \frac{d^nq}{q^2[(q-p)^2-m^2]}$$

einfachere Behandlung der Integrale bei einem Nenner

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax+B(1-x)]^2} = -\frac{1}{A-B}\left[\frac{1}{A}-\frac{1}{B}\right] = \frac{1}{AB}$$

$$= \int_0^1 dx \int_0^1 dy \, \frac{\delta_1(x+y-1)}{[Ax+By]^2}$$

$$\boxed{\frac{1}{\prod_{i=1}^N A_i} = \Gamma(N) \int_0^1 dx_1 \dots dx_N \, \frac{\delta_1\left[\sum x_i - 1\right]}{\left[\sum A_i x_i\right]^N}}$$

weitere Formeln durch Differentiation nach A_i

Grundintegrale der n-dimensionalen Regularisierung:

$$(*)\int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2+2kQ-M^2]^\alpha} = \frac{i(-1)^\alpha}{\Gamma(\alpha)(4\pi)^{n/2}} \frac{\Gamma(\alpha-\frac{n}{2})}{[Q^2+M^2]^{\alpha-\frac{n}{2}}}$$

$M^2 \equiv M^2 - i\varepsilon$

$\Gamma(x) \approx \frac{1}{x}$ für $x \to 0$

daraus durch Differentiation abzuleiten:

$$\int \frac{d^n k}{(2\pi)^n} \frac{k^\mu}{[k^2+2kQ-M^2]^\alpha} = \frac{i(-1)^{\alpha+1}}{\Gamma(\alpha)(4\pi)^{n/2}} \frac{\Gamma(\alpha-\frac{n}{2})}{[Q^2+M^2]^{\alpha-\frac{n}{2}}}\left(-Q^\mu\right)$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{k^\mu k^\nu}{[k^2+2kQ-M^2]^\alpha} = \frac{i(-1)^\alpha}{\Gamma(\alpha)(4\pi)^{n/2}}\left\{\frac{\Gamma(\alpha-\frac{n}{2})}{[Q^2+M^2]^{\alpha-\frac{n}{2}}} Q^\mu Q^\nu - \frac{\Gamma(\alpha-1-\frac{n}{2})}{[Q^2+M^2]^{\alpha-1-\frac{n}{2}}}\frac{g^{\mu\nu}}{2}\right\}$$

Oberfläche der n-dimensionalen Kugel: $\Omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(N) = (N-1)!$

[aus $\left\{\int_{-\infty}^{\infty} dx\, e^{-x^2}\right\}^n$ berechnet in kartesischen und Kugelkoordinaten]

Beweis (*):

$$\int d^n k = \int_{-\infty}^{+\infty} dk_0 \int_0^\infty d\omega\, \omega^{n-2} \int d\Omega_{n-1}$$

$$I_n(Q) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2+2kQ-M^2]^\alpha} = \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2-(Q^2+M^2)]^\alpha} \quad (k \to k-Q)$$

$$= \frac{2\pi^{-1/2}}{(4\pi)^{n/2}\Gamma(\frac{n-1}{2})} \int_{-\infty}^{+\infty} dk_0 \int_0^\infty \frac{d\omega\, \omega^{n-2}}{[k_0^2-\omega^2-(Q^2+M^2)]^\alpha}$$

Euler-Funktion: $B(x,y) = 2\int_0^\infty dt\, t^{2x-1}(1+t^2)^{-x-y} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

$$I_n(Q) = \frac{2\Gamma(\alpha-\frac{n-1}{2})}{(4\pi)^{n/2}\sqrt{\pi}\,\Gamma(\alpha)} \int_0^\infty dk_0 \frac{(-1)^\alpha}{[Q^2+M^2-k_0^2]^{\alpha-\frac{n-1}{2}}} = i\frac{(-1)^\alpha\Gamma(\alpha-\frac{n}{2})}{(4\pi)^{n/2}\Gamma(\alpha)[Q^2+M^2]^{\alpha-\frac{n}{2}}}$$

Clifford-Algebra in n Dimensionen:

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1} \qquad [\mathbb{1} = 4\text{-dim. Einheitsmatrix}]$$

möglich in n Dimensionen

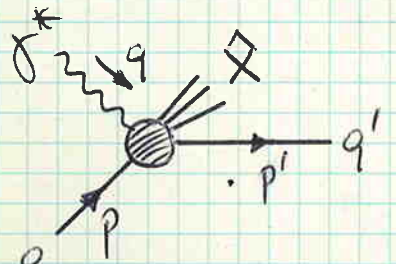
$$\longrightarrow \quad \mathrm{Tr}\, \gamma^\mu \gamma^\nu = 4 g^{\mu\nu}$$

$$\gamma^\mu \gamma_\mu = g^\mu{}_\mu = n$$

$$\gamma^\mu \not{a}\, \gamma_\mu = 2 a^\mu \gamma_\mu - \not{a}\, \gamma^\mu \gamma_\mu = (2-n) \not{a} \quad \text{etc.}$$

bis auf Anomalien kann γ_5 wie gewöhnlich behandelt werden $\Big\}$ $\{\gamma^\mu, \gamma_5\} = 0$, $\gamma_5^2 = \mathbb{1}$

Tiefinelastische lN-Streuung:



$$z = \frac{-q^2}{2pq} = \frac{Q^2}{2pq} \quad \longrightarrow \quad -\frac{pq}{q^2} = \frac{1}{2z}$$

$$q = p' - p + p_X$$

Partontensor: $$\hat{W}^{\mu\nu} = \hat{F}_1(z, Q^2)\left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right] + \frac{2z}{Q^2} \hat{F}_2(z, Q^2)\left(p^\mu + \frac{q^\mu}{2z}\right)\left(p^\nu + \frac{q^\nu}{2z}\right)$$

$$\longrightarrow \quad p^\mu p^\nu \hat{W}_{\mu\nu} = \frac{Q^2}{4z^2}\left(\frac{\hat{F}_2}{2z} - \hat{F}_1\right) \equiv \frac{Q^2}{4z^2} \frac{\hat{F}_L}{2z}$$

$$-g^{\mu\nu} \hat{W}_{\mu\nu} = (1-\varepsilon)\frac{\hat{F}_2}{z} - \frac{3-2\varepsilon}{2z} \hat{F}_L$$

$$\boxed{\begin{aligned} \hat{\mathfrak{F}}_1(z, Q^2) &= \hat{F}_1(z, Q^2) \\ \hat{\mathfrak{F}}_{2,L}(z, Q^2) &= \frac{\hat{F}_{2,L}(z, Q^2)}{2z} \end{aligned}}$$

Strukturfunktionen: $$\mathfrak{F}_i(x, Q^2) = \sum_{q,\bar{q}} e_q^2 \int_0^1 dy\, dz\, \hat{\mathfrak{F}}_i(z, Q^2)\, q(y, Q^2)\, \delta(x - yz)$$

1.) Bornterm: $$\mathcal{M}^\mu_{LO} = -i e e_q \delta_{ij}\, \bar{u}(p') \gamma^\mu u(p)$$

$$\hat{W}^{\mu\nu}_{LO} = \frac{1}{N_c} \sum \mathcal{M}^\mu_{LO} \mathcal{M}^{\nu *}_{LO} \frac{dPS_1(p+q; p')}{8\pi \sigma_0}$$

1-Teilchen-Phasenraum:

$$dPS_1(p+q;\, p') = \frac{d^{n-1}p'}{(2\pi)^{n-1}\,2p'^0}\,(2\pi)^n\,\delta_n(p+q-p') = 2\pi\, d^np'\,\delta_n(p+q-p')\,\delta_+(p'^2)$$

$$= 2\pi\,\delta\left[(p+q)^2\right] = \frac{2\pi}{Q^2}\,\delta(1-z)$$

$$\hat W^{\mu\nu}_{LO} = e^2 e_q^2\,\frac{N_c}{N_c}\,\mathrm{Tr}\left(\not p'\gamma^\mu \not p\,\gamma^\nu\right)\frac{1}{8\pi}\,\frac{2\pi}{Q^2}\,\delta(1-z)\,\frac{1}{\sigma_0}$$

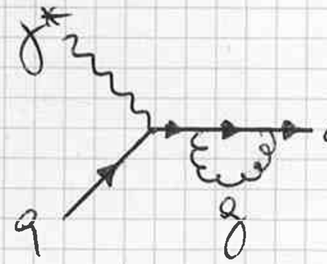
$$= \frac{2\pi\alpha e_q^2}{\sigma_0}\left\{\left[-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right]+\frac{4}{Q^2}\left(p^\mu+\frac{q^\mu}{2}\right)\left(p^\nu+\frac{q^\nu}{2}\right)\right\}\delta(1-z)$$

$$\Longrightarrow \boxed{\hat F_{L,LO}=0} \qquad \boxed{\hat F_{1,LO} = \hat F_{2,LO} = \delta(1-z)}$$

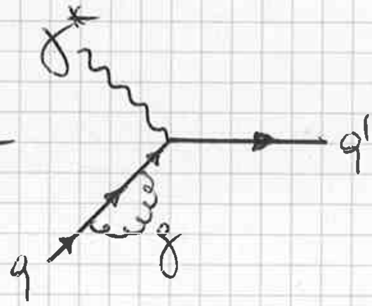
$$\sigma_0 = 2\pi\alpha e_q^2$$

2.) QCD-Korrekturen:

(a) virtuelle Korrekturen:



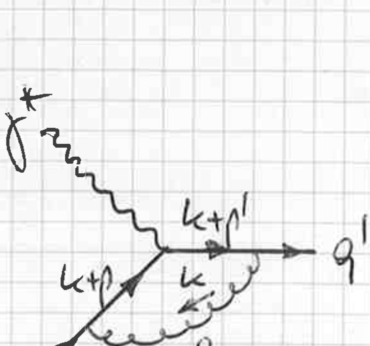
$+$



$\equiv 0$ [Partonen masselos]

denn: $\displaystyle\int \frac{d^nk}{(2\pi)^n}\,\frac{1}{(k^2)^\alpha} = 0$ $\qquad \displaystyle\int \frac{d^nk}{(2\pi)^n}\,\frac{1}{k^2(k+p)^2} \propto \frac{(-p^2)^{-\varepsilon}}{\varepsilon} \to 0$

für $\varepsilon < 0$
[analyt. Fortsetzung]



$$M_V = i^6(-1)^4\, e e_q\, \bar g_s^2\,(T^aT^a)_{ij}\; *$$
$$*\;\bar u(p')\int\frac{d^nk}{(2\pi)^n}\,\frac{\gamma^\alpha(\not k+\not p')\gamma^\mu(\not k+\not p)\gamma_\alpha}{k^2(k+p)^2(k+p')^2}\,u(p)$$

dimensionslose Kopplung: $\bar g_s^2 = g_s^2\,\mu^{2\varepsilon}$

$$\hat{W}^{\mu\nu}_{V} = \frac{1}{N_C} \sum 2\,\mathrm{Re}\, \mathcal{M}^{\mu}_{V} \mathcal{M}^{\nu *}_{LO} \frac{dPS_1(p+q;p')}{8\pi\, \sigma_0}$$

$$\Rightarrow \delta\hat{\mathcal{F}}_{4V} = 0$$

$$\delta\hat{\mathcal{F}}_{2,V} = \delta\hat{\mathcal{F}}_{1,V} = 2C_F i g_S^2 \left\{ (3+2\varepsilon) B_0(q;0,0) - 2Q^2 C_0(p,p';0,0,0) \right\} \delta(1-z)$$

$$B_0(q;0,0) = \mu^{2\varepsilon} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2(k+q)^2} = i \frac{\Gamma(\varepsilon)}{(4\pi)^2} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \underbrace{\int_0^1 dx\, x^{-\varepsilon}(1-x)^{-\varepsilon}}_{= B(1-\varepsilon,1-\varepsilon) = \frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)}}$$

$$= i \frac{\Gamma(1+\varepsilon)}{(4\pi)^2} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \frac{1}{\varepsilon(1-2\varepsilon)}$$

Gammafunktion: $\Gamma(1+\varepsilon) = \exp\left\{ -\gamma_E \varepsilon + \sum_{i=2}^{\infty} \frac{(-1)^i}{i} \zeta(i) \varepsilon^i \right\}$

Eulerkonstante $\gamma_E = 0.577215\ldots$

Riemann'sche Zetafkt.

$\zeta(2) = \frac{\pi^2}{6}$

$\zeta(3) = 1.202056\ldots$

$\zeta(4) = \frac{\pi^4}{90}$

$\vdots$

$$\Rightarrow \Gamma(1+\varepsilon)\,\Gamma(1-\varepsilon) = 1 + \varepsilon^2 \zeta(2) + O(\varepsilon^4)$$

$$\underline{\underline{B_0(q;0,0) = \frac{i}{(4\pi)^2} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \left(\frac{1}{\varepsilon} + 2\right) + O(\varepsilon)}}$$

$$C_0(p,p';0,0,0) = \mu^{2\varepsilon} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2(k+p)^2(k+p')^2}$$

$$= -i \frac{\Gamma(1+\varepsilon)}{(4\pi)^2 Q^2} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \underbrace{\int_0^1 dx\,dy\, y^{-1-\varepsilon} x^{-1-\varepsilon} (1-x)^{-\varepsilon}}$$

divergent für $\varepsilon > 0$ $(n<4)$

konvergent für $\varepsilon < 0$ $(n>4)$

analytische Fortsetzung: $\int_0^1 dz\, z^{x-1}(1-z)^{y-1} \equiv B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Infrarote und kollineare Singularitäten werden für $n>4$ ($\varepsilon<0$) regularisiert durch analytische Fortsetzung.

$$\Rightarrow \int_0^1 dy\, y^{-1-\varepsilon} = \frac{\Gamma(-\varepsilon)\Gamma(1)}{\Gamma(1-\varepsilon)} = -\frac{1}{\varepsilon}$$

$$\int_0^1 dx\, x^{-1-\varepsilon}(1-x)^{-\varepsilon} = \frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} = -\frac{1}{\varepsilon}\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$C_0(p_1,p_1';0,0,0) = -i\,\frac{\Gamma(1+\varepsilon)}{(4\pi)^2 Q^2}\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\frac{1}{\varepsilon^2}$$ IR, COLL

$$= \frac{i}{(4\pi)^2 Q^2}\,\frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\left[-\frac{1}{\varepsilon^2}-\zeta(2)\right]+O(\varepsilon)$$

Renormierung: $m=0 \Rightarrow \delta m=0$

$$M^\mu_{CT} = M^\mu_{LO}\left\{\sqrt{Z_2}^{\,2}\,\frac{\sqrt{Z_3}}{Z_1}-1\right\}$$

Ward-Identität: $Z_1 = Z_2$

Photon-Propagator: $Z_3 = 1$

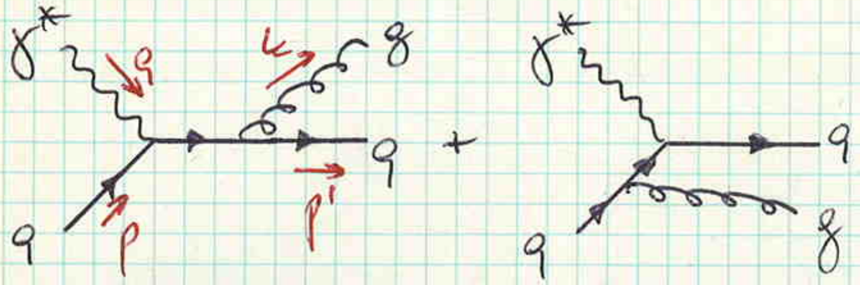
$\Rightarrow M_{CT} = 0$ keine Renormierung nach Summation aller Diagramme

$$\boxed{\delta\hat{\mathcal{F}}_{2,V} = \delta\hat{\mathcal{F}}_{1,V} = C_F\frac{\alpha_s}{2\pi}\frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\left[-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}-8-2\zeta(2)\right]\delta(1-z)}$$

$$\boxed{\delta\hat{\mathcal{F}}_{4V} = 0}$$

Infrarote Divergenzen werden durch Addition der reellen Gluonabstrahlung subtrahiert.

(ii) reelle Korrekturen:



$$s=(p+q)^2,\quad t=(p'-p)^2,\quad u=(k-p)^2 \quad\Rightarrow\quad s+t+u=q^2$$

$$M_q^\mu = i^3(-1)^2 e\, e_q\, g_s\, T^a_{ij}\, \bar{u}(p')\left\{\frac{\gamma^\alpha(p'+k)\gamma^\mu}{(p'+k)^2}+\frac{\gamma^\mu(p-k)\gamma^\alpha}{(p-k)^2}\right\}u(p)\,\varepsilon^*_\alpha$$

$$\hat{W}^{\mu\nu}_{\gamma q}=\frac{1}{N_c}\int\sum M_q^\mu M_q^{\nu *}\,\frac{dPS_2(p+q;p',k)}{8\pi\sigma_0}$$

$$dPS_2=\frac{d^{n-1}p'}{(2\pi)^{n-1}2p'^0}\,\frac{d^{n-1}k}{(2\pi)^{n-1}2k^0}(2\pi)^n\,\delta_n(p+q-p'-k)$$

$$=\frac{d^{n-1}k}{(2\pi)^{n-2}2k^0}\,d^np'\,\delta_+(p'^2)\,\delta_n(p+q-p'-k)$$

$$=\frac{|\vec{k}|^{n-2}\,d|\vec{k}|\,d\Omega_{n-1}}{2(2\pi)^{n-2}k^0}\,\delta\left[(p+q-k)^2\right]$$

c.m.s: $p+q=\sqrt{s}(1,\vec{0}) \longrightarrow (p+q-k)^2=s-2\sqrt{s}\,k^0$

$0=k^2=(k^0)^2-|\vec{k}|^2 \longrightarrow k^0dk^0=|\vec{k}|\,d|\vec{k}|$

$$dPS_2=\frac{(k^0)^{n-3}dk^0\,d\Omega_{n-1}}{2(2\pi)^{n-2}}\,\delta(s-2\sqrt{s}\,k^0)=\frac{s^{\frac{n-4}{2}}}{2^{2n-3}\pi^{n-2}}\,d\Omega_{n-1}$$

Oberflächenintegral: ϑ = Streuwinkel des Quarks

$$d\Omega_{n-1}=\sin^{n-3}\vartheta\,d\vartheta\,d\Omega_{n-2}=2\frac{\pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-2}{2}\right)}\left(1-\cos^2\vartheta\right)^{\frac{n-4}{2}}d\cos\vartheta$$

Substitution: $y=\frac{1}{2}(1+\cos\vartheta)$

$$\boxed{dPS_2 = \frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^{\varepsilon} \frac{y^{-\varepsilon}(1-y)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\,dy} \qquad 0 \le y \le 1$$

Parametrisierung: $p = \frac{s+Q^2}{2\sqrt{s}}(1, 0, \vec{0}, 1) \qquad q = \left(\frac{s-Q^2}{2\sqrt{s}}, 0, \vec{0}, -\frac{s+Q^2}{2\sqrt{s}}\right)$

$p' = \frac{\sqrt{s}}{2}(1, \sin\vartheta, \vec{0}, \cos\vartheta) \qquad k = \frac{\sqrt{s}}{2}(1, -\sin\vartheta, \vec{0}, -\cos\vartheta)$

$$\longrightarrow \quad s = \frac{1-z}{z}Q^2; \qquad t = -\frac{Q^2}{z}(1-y); \quad u = -\frac{Q^2}{z}y$$

$$\boxed{\delta\hat{\mathcal{F}}_{Lq} = \frac{4z^2}{Q^2} p_\mu p_\nu \hat{W}^{\mu\nu}_{\gamma q} = \int_0^1 dy\, \frac{4}{3}\frac{\alpha_s}{2\pi} 4z^2 \frac{-t}{Q^2} = \frac{4}{3}\frac{\alpha_s}{2\pi} 2z \neq 0} \leftarrow \text{endlich}$$

$$\delta\hat{\mathcal{F}}_{2q} - \frac{3}{2}\delta\hat{\mathcal{F}}_{Lq} = \frac{-g_{\mu\nu}\hat{W}^{\mu\nu}_{\gamma q}}{2(1-\varepsilon)} + O(\varepsilon)$$

$$= \frac{4}{3}\frac{\alpha_s}{2\pi}\frac{z^\varepsilon(1-z)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\int_0^1 dy\, y^{-\varepsilon}(1-y)^{-\varepsilon}\underbrace{\left\{(1-\varepsilon)\left(\frac{s}{-u} + \frac{-u}{s}\right) + 2\frac{tQ^2}{su} + 2\varepsilon\right\}}$$

$$= (1-\varepsilon)\left(\frac{1-z}{y} + \frac{y}{1-z}\right) + 2\frac{(1-y)z}{y(1-z)} + 2\varepsilon$$

$$= \frac{4}{3}\frac{\alpha_s}{2\pi}\frac{z^\varepsilon(1-z)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\left\{(1-\varepsilon)\left[\frac{1-z}{-\varepsilon} + \frac{1-\varepsilon}{(2-2\varepsilon)(1-2\varepsilon)(1-z)}\right] - 2\frac{1-\varepsilon}{\varepsilon(1-2\varepsilon)}\frac{z}{1-z}\right.$$

$$\left. + \frac{2\varepsilon}{1-2\varepsilon}\right\}\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

Distributionen: $(1-z)^{-1-\varepsilon} = \left(\frac{1}{(1-z)^{1+\varepsilon}}\right)_+ + \delta(1-z)\int_0^1 \frac{dz'}{(1-z')^{1+\varepsilon}}$

$$= -\frac{1}{\varepsilon}\delta(1-z) + \left(\frac{1}{1-z}\right)_+ - \varepsilon\left(\frac{\log(1-z)}{1-z}\right)_+ + O(\varepsilon^2)$$

$$\int_a^1 dz\, \frac{f(z)}{(1-z)_+} = \int_a^1 dz\, \frac{f(z)-f(1)}{1-z} - \int_0^a \frac{dz}{1-z} f(1)$$

$$\delta \hat{\mathcal{F}}^{1}_{2q} - \frac{3}{2}\delta \hat{\mathcal{F}}^{1}_{Lq} = C_F \frac{\alpha_s}{2\pi}\frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\Bigg\{\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{7}{2}\right]\delta(1-z)$$

$$-\left(\frac{1}{\varepsilon}+\log z\right)\left(\frac{1+z^2}{1-z}\right)_+ + (1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+ - \frac{3}{2}\left(\frac{1}{1-z}\right)_+ + 3 - z\Bigg\}$$

- Altarelli–Parisi–Splittingfkt:

$$P_{qq}(z) = C_F\frac{1+z^2}{1-z} - \delta(1-z)\int_0^1 dz'\, C_F \frac{1+z'^2}{1-z'}$$

$$\Longrightarrow \int_0^1 dz\, f(z)\, P_{qq}(z) = \int_0^1 dz\, C_F\frac{1+z^2}{1-z}\left[f(z)-f(1)\right] = \int_0^1 dz\, C_F\left(\frac{1+z^2}{1-z}\right)_+ f(z)$$

$$\Longrightarrow \boxed{P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+ = C_F\left\{\left(\frac{2}{1-z}\right)_+ - 1 - z + \frac{3}{2}\delta(1-z)\right\}}$$

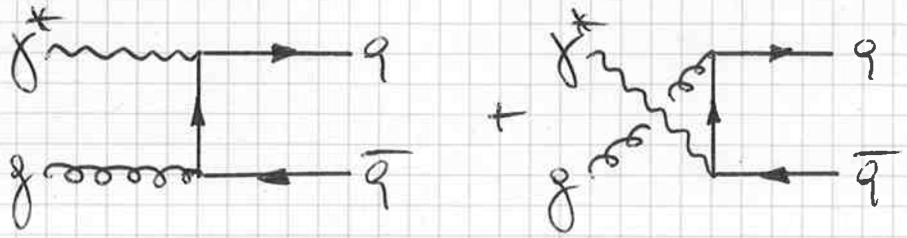
- Summe virtuell + reell:

$$\delta \hat{\mathcal{F}}^{1\,\gamma q}_{2} = \delta\hat{\mathcal{F}}^{1}_{2V} + \delta\hat{\mathcal{F}}^{1}_{2q}$$

$$= \frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\frac{\alpha_s}{2\pi}\left[-\frac{1}{\varepsilon}-\log z\right]P_{qq}(z) + C_F\frac{\alpha_s}{2\pi}\Bigg\{(1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+ - \frac{3}{2}\left(\frac{1}{1-z}\right)_+$$

$$+3+2z-\left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\Bigg\}$$

Der gekreuzte Kanal $\gamma^* g \to q\bar{q}$ ist von gleicher Ordnung in α_s und von $\gamma^* q \to qg$ nicht unterscheidbar.



$$\delta\hat{\mathcal{F}}_L^{\gamma g} = T_R \frac{\alpha_s}{2\pi} 4z(1-z) \qquad \left[T_R = \frac{1}{2}\right]$$ 60

$$\delta\hat{\mathcal{F}}_2^{\gamma g} = \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\left(-\frac{1}{\varepsilon} - \log\frac{z}{1-z}\right)\frac{\alpha_s}{2\pi}P_{qg}(z) + T_R\frac{\alpha_s}{2\pi}\left[8z(1-z)-1\right]$$

$$P_{qg}(z) = T_R\left[z^2 + (1-z)^2\right]$$

- Gluon-Spin-Mittelung: $\frac{1}{2} \longrightarrow \frac{1}{n-2} = \frac{1}{2(1-\varepsilon)}$

- DIS-Strukturfunktion:

$$F_2(x,Q^2) = 2x\sum_q e_q^2\left\{\left[q_0 + \bar{q}_0\right]\otimes\hat{\mathcal{F}}_2^{\gamma q} + g_0\otimes\hat{\mathcal{F}}_2^{\gamma g}\right\}$$

mit der Faltung:

$$f\otimes g = \int_0^1 dy\,dz\, f(y)g(z)\delta(x-yz) = \int_x^1 \frac{dz}{z} f\left(\frac{x}{z}\right)g(z)$$

Die verbleibenden kollinearen Singularitäten werden durch die Renormierung der Partondichten beseitigt.

Renormierung der Partondichten [Massenfaktorisierung]

$$q_0(x) = \Gamma_{qq}\otimes q(x,\mu_F^2) + \Gamma_{qg}\otimes g(x,\mu_F^2)$$

$$\Gamma_{ij}(x) = \delta_{ij}\delta(1-x) + \frac{\alpha_s}{2\pi}\left\{\frac{1}{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\varepsilon}P_{ij}(x) - f_{ij}(x)\right\}$$

μ_F = Faktorisierungsskala der Partondichten

$$\Rightarrow \mu_F^2\frac{\partial q_0(x)}{\partial\mu_F^2} = 0 = -\frac{\alpha_s}{2\pi}\left[P_{qq}\otimes q(x,\mu_F^2) + P_{qg}\otimes g(x,\mu_F^2)\right] + \mu_F^2\frac{\partial q(x,\mu_F^2)}{\partial\mu_F^2} + O(\alpha_s^2)$$

$\Rightarrow q(x,\mu_F^2)$ ist Lösung der Altarelli-Parisi-Glgn. in LO

Ergebnis:

$$\boxed{\begin{aligned}
F_2(x,Q^2) &= 2x\sum_q e_q^2\left[q(x,\mu_F^2)+\bar q(x,\mu_F^2)\right]+\Delta F_2(x,Q^2)\\
\Delta F_2(x,Q^2) &= 2x\,\frac{\alpha_s}{2\pi}\sum_q e_q^2\int_x^1\frac{dz}{z}\Big\{C_F\left[q\left(\tfrac{x}{z},\mu_F^2\right)+\bar q\left(\tfrac{x}{z},\mu_F^2\right)\right]\ast\\
&\quad\ast\Big[-\frac{P_{qq}(z)}{C_F}\log\frac{\mu_F^2 z}{Q^2}+(1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+-\frac{3}{2}\frac{1}{(1-z)_+}+3+2z\\
&\qquad-\left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)-\frac{f_{qq}(z)}{C_F}\Big]\\
&\quad+T_R\, g\left(\tfrac{x}{z},\mu_F^2\right)\Big[-\frac{P_{qg}}{T_R}\log\frac{\mu_F^2 z}{Q^2(1-z)}+8z(1-z)-1\Big]-\frac{f_{qg}(z)}{T_R}\Big\}\\
F_L(x,Q^2) &= F_2(x,Q^2)-2xF_1(x,Q^2)\\
&= 2x\,\frac{\alpha_s}{2\pi}\sum_q e_q^2\int_x^1\frac{dz}{z}\Big\{C_F\left[q\left(\tfrac{x}{z},\mu_F^2\right)+\bar q\left(\tfrac{x}{z},\mu_F^2\right)\right]2z+T_R\, g\left(\tfrac{x}{z},\mu_F^2\right)4z(1-z)\Big\}
\end{aligned}}$$

PHYSIKALISCHE INTERPRETATION

1.) natürliche Faktorisierungsskala: $\mu_F^2 = Q^2$

2.) Faktorisierungsschemata:

(i) $\overline{MS}$-Schema: $f_{ij}^{\overline{MS}}(z)\equiv 0$

(ii) DIS-Schema: $F_2(x,Q^2)\equiv 2x\sum_q e_q^2\left[q(x,Q^2)+\bar q(x,Q^2)\right]$

$$\Longrightarrow \Delta F_2 \equiv 0 \quad \left[\mu_F^2=Q^2\right]$$

$$\Longrightarrow f_{qq}^{DIS}(z) = C_F\Big\{-\frac{1+z^2}{1-z}\log z+(1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+-\frac{3}{2}\frac{1}{(1-z)_+}+3+2z-\left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\Big\}$$

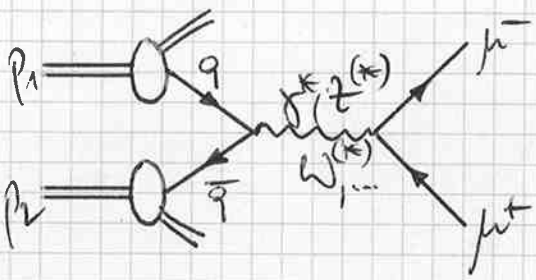
$$f_{qg}^{DIS}(z) = T_R\Big\{\left[z^2+(1-z)^2\right]\log\frac{1-z}{z}+8z(1-z)-1\Big\}$$

Faktorisierungstheorem der QCD

Partonische Wirkungsquerschnitte weisen kollineare Divergenzen im hadronischen Anfangszustand auf, die universell [prozeßunabhängig] vom harten Streuprozeß faktorisieren und in den renormierten Partondichten des Anfangszustands absorbiert werden können. Diese renormierten Partondichten sind Lösungen der Altarelli-Parisi-Gleichungen.

§5. Drell-Yan-Prozesse

Produktion elw. wechselwirkender Teilchen in Hadronkollisionen:



$$p\bar{p} \to \mu^+\mu^- + X$$
$$p\bar{p} \to W^\pm, Z + X \quad \ldots$$

WQ: $$\sigma(p_1 p_2 \to \mu^+\mu^- + X) = \sum_q \int_0^1 dx_1\, dx_2 \left[q(x_1)\bar{q}(x_2) + \bar{q}(x_1) q(x_2)\right] \hat{\sigma}\left[q\bar{q} \to \mu^+\mu^-; \hat{s}\right]$$

invariante Energie: $\hat{s} = (p_q + p_{\bar{q}})^2 = 2 p_q p_{\bar{q}} = x_1 x_2 * 2 p_1 p_2 = x_1 x_2 s$

Parton-WQ: $\hat{\sigma}(q\bar{q} \to \mu^+\mu^-) = \left(\frac{1}{N_c}\right)^2 N_c \frac{4\pi\alpha^2}{3\hat{s}} e_q^2 = \frac{e_q^2}{N_c} \frac{4\pi\alpha^2}{3\hat{s}}$

Integrationsgrenzen: $\hat{s} = x_1 x_2 s \geq (2m_\mu)^2$

$$\Longrightarrow x_2 \geq \frac{\tau_0}{x_1} \; ; \; x_1 \geq \tau_0 = \frac{4m_\mu^2}{s}$$

$$\sigma = \int_{\tau_0}^1 d\tau \left[\sum_q \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2 \left[q(x_1)\bar{q}(x_2) + \bar{q}(x_1) q(x_2)\right] \delta(\tau - x_1 x_2)\right] \hat{\sigma}(\hat{s} = \tau s)$$

$$= \int_{\tau_0}^{1} d\tau \sum_q \int_{\tau}^{1} \frac{dx}{x}\left[q(x)\,\bar q\left(\tfrac{\tau}{x}\right) + \bar q(x)\, q\left(\tfrac{\tau}{x}\right)\right] \hat\sigma(\tau s)$$

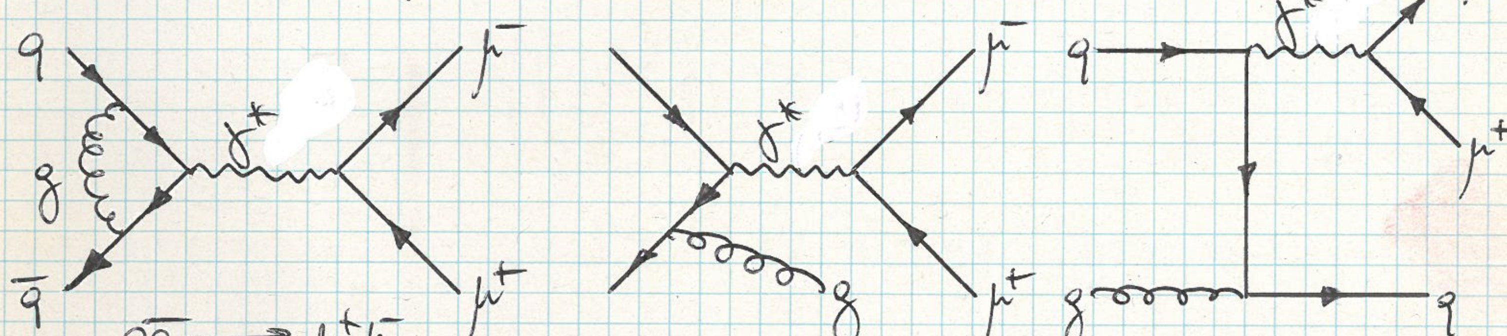
$$\sigma = \int_{\tau_0}^{1} d\tau \sum_q \frac{d\mathcal{L}^{q\bar q}}{d\tau}\, \hat\sigma(\hat s = \tau s)$$

$\uparrow$

„Luminosität von $q, \bar q$ in Hadronstrahlen"

$$M^4_{\mu\mu}\frac{d\sigma}{dM^2_{\mu\mu}} = \frac{4\pi\alpha^2}{3N_c}\,\tau_\mu \sum_q e_q^2 \frac{d\mathcal{L}^{q\bar q}}{d\tau_\mu}\Bigg|_{\tau_\mu = \frac{M^2_{\mu\mu}}{s}}$$

QCD – Strahlungskorrekturen:



$q\bar q \xrightarrow[\text{virt}]{} \mu^+\mu^-$ $\qquad$ $q\bar q \to \mu^+\mu^- + g$ $\qquad$ $qg \to \mu^+\mu^- + q$

$$M^4_{\mu\mu}\frac{d\sigma}{dM^2_{\mu\mu}} = \frac{4\pi\alpha^2}{3N_c}\,\tau_\mu \int_{\tau_0}^{1}\frac{d\tau}{\tau}\Bigg\{\sum_q e_q^2\frac{d\mathcal{L}^{q\bar q}}{d\tau}\left[\delta(1-z) + \frac{\alpha_s(\mu_R^2)}{\pi}D_{qq}(z)\right]$$

$$+ \sum_{q,\bar q} e_q^2 \frac{d\mathcal{L}^{gq}}{d\tau}\,\frac{\alpha_s(\mu_R^2)}{\pi}\,D_{gq}(z)\Bigg\} \qquad \left[z = \frac{\tau_\mu}{\tau}\right]$$

$$D_{qq}(z) = -P_{qq}(z)\log\frac{\mu_F^2 z}{M^2_{\mu\mu}} + C_F\left\{2\left[\frac{\pi^2}{6}-2\right]\delta(1-z) + 2(1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+\right\}$$

$$- f_{qq}(z)$$

$$D_{gq}(z) = -\frac{1}{2}P_{qg}(z)\log\frac{\mu_F^2 z}{M^2_{\mu\mu}(1-z)^2} + \frac{T_R}{4}(1+6z-7z^2) - f_{qg}(z)$$

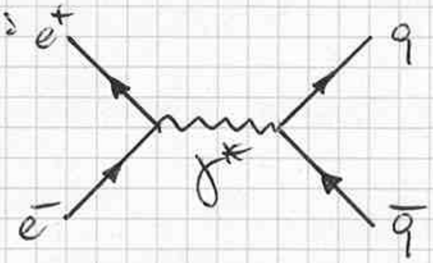
$$\frac{d\mathcal{L}^{gq}}{d\tau} = \int_{\tau}^{1}\frac{dx}{x}\left[q(x,\mu_F^2)\,g\left(\tfrac{\tau}{x},\mu_F^2\right) + g(x,\mu_F^2)\,q\left(\tfrac{\tau}{x},\mu_F^2\right)\right]$$

Bemerkung: In analoger Weise können aufgrund des Faktorisierungstheorems behandelt werden:

$$p_1 p_2 \to W^{\pm}, Z + X$$
$$p_1 p_2 \to t\bar{t} + X$$
$$p_1 p_2 \to nj + X$$

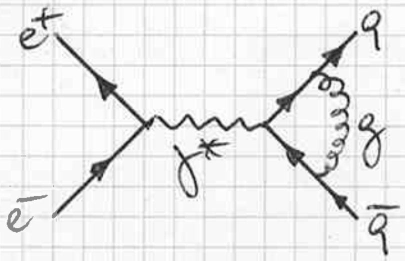
§6. $e^+e^- \to$ Hadronen: totales WQ

Partonbild:

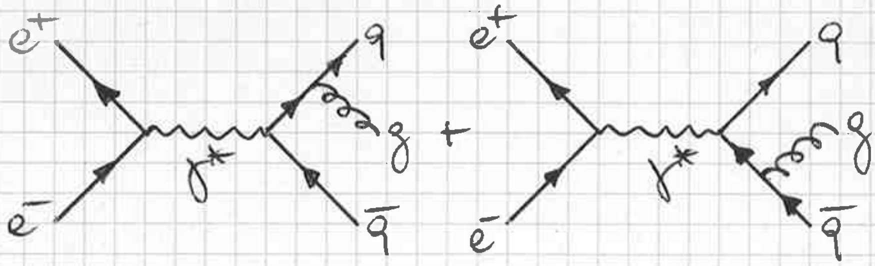


$$\sigma_0 = \frac{4\pi\alpha^2}{3s} N_c e_q^2$$

QCD: virtuelle Korr.



reelle Korr.



$$\sigma = (1 + \delta_V + \delta_R)\,\sigma_0$$

$$\delta_V = C_F \frac{\alpha_s}{\pi} \Gamma(1+\varepsilon) \left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon} \left\{ -\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4 + \frac{2}{3}\pi^2 + O(\varepsilon) \right\}$$

$$\delta_R = C_F \frac{\alpha_s}{\pi} \Gamma(1+\varepsilon) \left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon} \left\{ \frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{19}{4} - \frac{2}{3}\pi^2 + O(\varepsilon) \right\}$$

totales WQ:

$$\sigma = \left(1 + \frac{3}{4} C_F \frac{\alpha_s}{\pi}\right)\sigma_0 = \left(1 + \frac{\alpha_s}{\pi}\right)\sigma_0$$

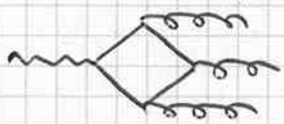
R-Wert $= \dfrac{\sigma(e^+e^- \to \text{had})}{\sigma(e^+e^- \to \mu^+\mu^-)}$: $R_B = 3\sum_q e_q^2$

$$R = R_B\left[1 + \frac{\alpha_s}{\pi} + \sum_{n\geq 2} c_n \left(\frac{\alpha_s}{\pi}\right)^n\right] \quad \text{für } \overline{MS} \text{ @ } \mu^2 = s$$

$$c_2 = \frac{365}{24} - 11\,\zeta(3) - \left(\frac{11}{12} - \frac{2}{3}\zeta(3)\right)N_F \approx 1.986 - 0.115\,N_F$$

$$c_3 = \frac{87029}{288} - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)$$

$$- \left(\frac{7847}{216} - \frac{262}{9}\zeta(3) + \frac{25}{9}\zeta(5)\right)N_F \qquad \eta = \frac{\left(\sum e_q\right)^2}{3\sum e_q^2}$$



$$+ \left(\frac{151}{162} - \frac{19}{27}\zeta(3)\right)N_F^2$$

$$- \frac{1}{72}\zeta(2)(33 - 2N_F)^2 + \eta\left(\frac{55}{72} - \frac{5}{3}\zeta(3)\right) \approx -6.637 - 1.200\,N_F - 0.005\,N_F^2 - 1.240\,\eta$$

Hochpräzisionsbestimmung von $\alpha_s(M_Z^2)^{\overline{MS}}_{(5)} = 0.122 \pm 0.003$

§7. Jets in der QCD

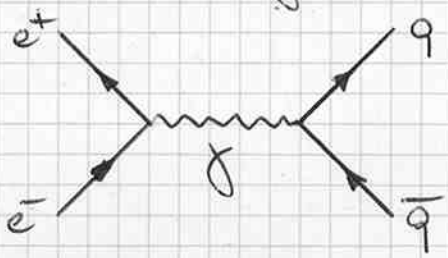
Asymptotische Freiheit: Im Femto-Universum $d \lesssim 10^{-15}$ cm laufen stark wechselwirkende Prozesse als Ein-Quanten-Prozesse auf Quark-Gluon-Niveau ab. [$\to$ analog e und γ in QED]

Jet-Hypothese: Partonkonfigurationen, im Femto-Universum ausgebildet, transformieren sich bei großen Abständen $d \gtrsim 10^{-13}$ cm in Büschel von Hadronen mit beschränktem Relativ-Transversalimpuls $p_\perp \lesssim 500$ MeV $\equiv$ Jets

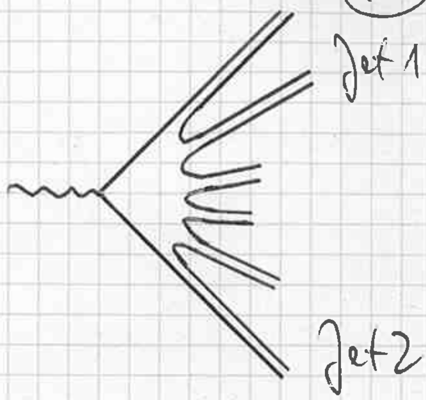
$\Rightarrow$ Jet-Analysen: Tests der QCD im Femto-Universum

Jet-Struktur: bestimmt durch (nicht-)perturbative QCD

(a) 0^{te} Ordnung QCD: $e^+e^- \to q\bar{q}$

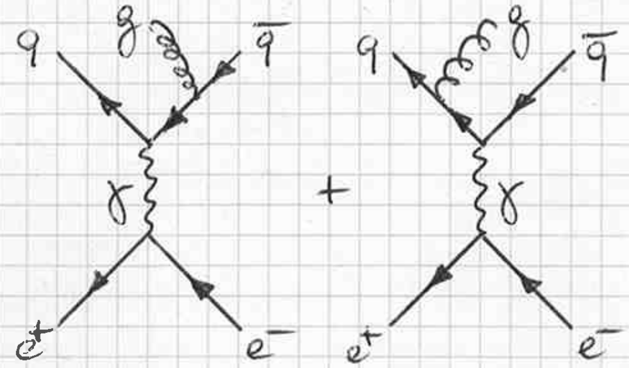


Energie-Fluß-röhre: spont. $q\bar{q}$ Produktion $\Rightarrow$ Aufbrechen der Röhre mit kleinem $p_\perp$



"SPEAR"-Jets

(b) Gluon-Jets in der e^+e^--Annihilation:



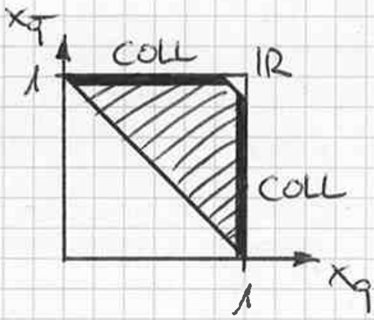
Beschleunigung von Color-Ladung $\Rightarrow$ Abstrahlung von gluonischen Eichquanten [$\sim \gamma$ Abstrahlung von beschleun. el. Ldg.]

$q\bar{q}g$ Kinematik: $x_i = \frac{E_i}{\sqrt{s}/2}$

mit $x_q + x_{\bar{q}} + x_g = 2$

$0 \le x_i < 1$

$x_q + x_{\bar{q}} = 2 - x_g \ge 1$



Dalitz-Plot

Pol für $(q+g)^2 = (Q-\bar{q})^2 = Q^2 - 2Q\bar{q} = Q^2(1-x_{\bar{q}})$ $\quad [Q = e^+ + e^- = \sqrt{s}(1,\vec{0})]$

$(\bar{q}+g)^2 = Q^2(1-x_q)$

$$\boxed{\frac{1}{\sigma_{q\bar{q}}}\frac{d\sigma}{dx_q\,dx_{\bar{q}}} = \frac{2}{3}\frac{\alpha_s}{\pi}\frac{x_q^2 + x_{\bar{q}}^2}{(1-x_q)(1-x_{\bar{q}})}}$$

$x_q \to 1$: $g \parallel \bar{q}$ kollin. Config.

$x_{\bar{q}} \to 1$: $g \parallel q$

$x_q, x_{\bar{q}} \to 1$: $x_g \to 0$ infrarote Config.

Experimentelle Entwicklung: Erhöhung der Energie $\to$

- Jets werden breiter
- klare 3-Jet-Events: PETRA-Jets

$\Leftarrow$ sichtbare QCD-Eichquanten

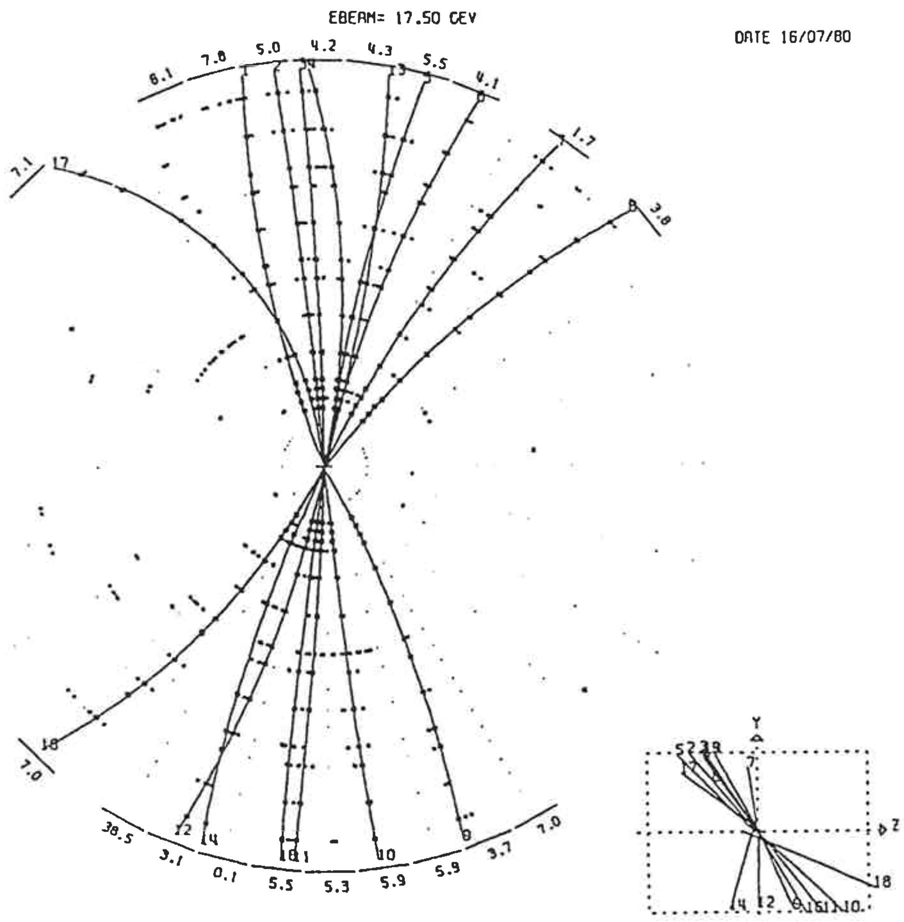


Fig. 29 A two-jet events as observed at W = 35 GeV in the TASSO detector.

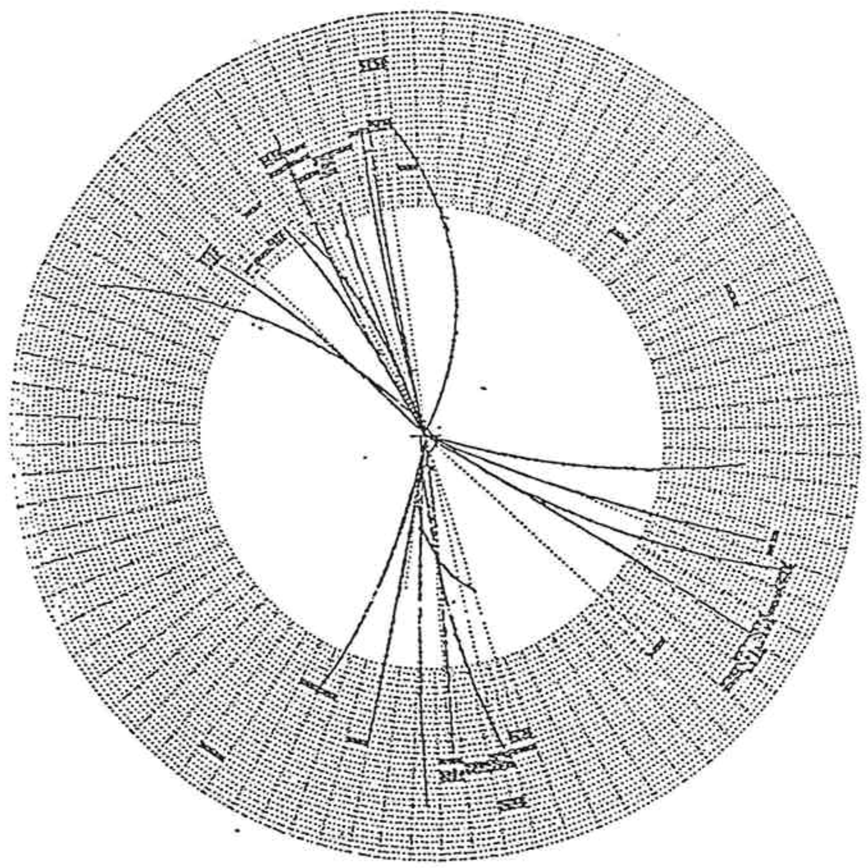


Fig. 11.12 A three-jet event observed by the JADE detector at PETRA.

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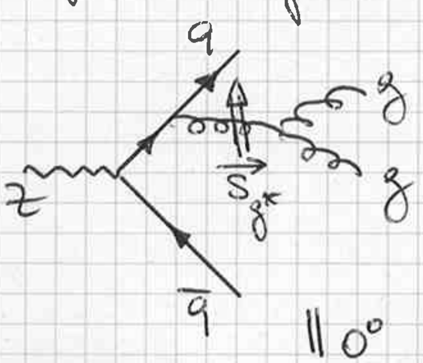
• Messung des Gluon-Spins:

$$s_g = 1 \Longrightarrow \rho_1 \sim \frac{1}{(1-x_q)(1-x_{\bar q})}$$ divergent für $x_g \to 0$; $x_q, x_{\bar q} \to 1$

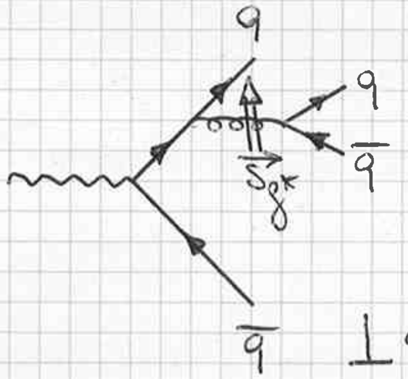
$$s_g = 0 \Longrightarrow \rho_0 \sim \frac{x_g^2}{(1-x_q)(1-x_{\bar q})}$$ endlich für $x_g \to 0$

• Messung der Gluon-Farbe:

4 jet-Ereignisse:



$\parallel 0°$



$\perp 90°$

$\chi = \sphericalangle(E_{12}, E_{34})$:

$$gg = \frac{(1-z+z^2)^2}{z(1-z)} + z(1-z)\cos 2\chi$$

$$q'\bar{q}' = \frac{1}{2}\left[z^2+(1-z)^2\right] - z(1-z)\cos 2\chi$$

SU_3: 0° vs. U_1: 90°

• Jet-Multiplizität:

$f_n(y)$ = Bruchteil von Ereignissen mit n Jets im Endzustand:

$$\sum f_n(y) = 1$$

y = max. Jet-Masse: $M^2_{jet} \le ys$

$$f_{n+2}(y) = \left(\frac{\alpha_s}{2\pi}\right)^n \sum_{j=0}^{\infty} C_{nj}(y) \left(\frac{\alpha_s}{2\pi}\right)^j$$ $\Longrightarrow$ Messung von α_s

Bsp: 2- und 3-Jet-Verteilungen:

$$f_3 = \int_{(p_i+p_j)^2 \ge ys} dx_1 dx_2 \, \frac{2}{3}\frac{\alpha_s}{\pi} \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

$$= \frac{2}{3}\frac{\alpha_s}{\pi}\left[(3-6y)\log\frac{y}{1-2y} + 2\log^2\frac{y}{1-y} + \frac{5}{2} - 6y - \frac{9}{2}y^2 + 4\,\mathrm{Li}_2\left(\frac{y}{1-y}\right) - \frac{\pi^2}{3}\right]$$

$$f_2 = 1 - f_3$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dy}{y}\log(1-y) = \sum \frac{x^n}{n^2} \text{ für } |x| \le 1$$

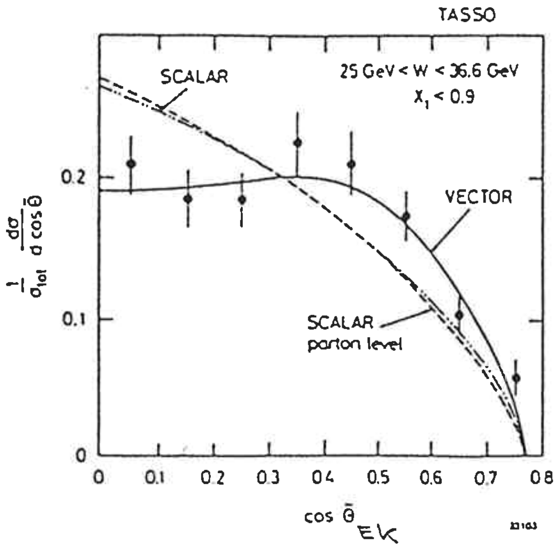


Fig. 7.25

The cosθ distribution of events with $x_1 < 0.9$. The solid line and the dashed-dotted line show the distribution predicted for vector gluons and scalar gluons respectively. The predictions include hadronization. For comparison the prediction for a scalar gluon on the parton level is shown in Fig. 7.24. The distributions are normalized to the number of observed events. EK = Ellis - Karliner

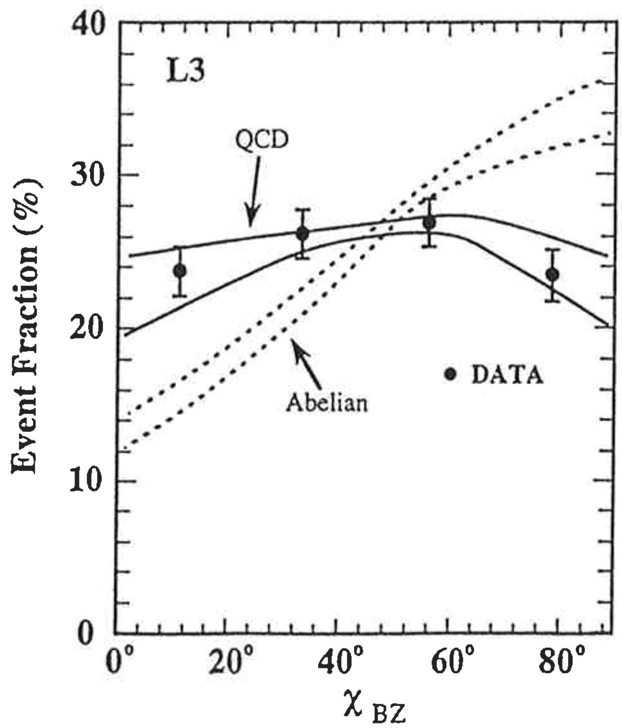
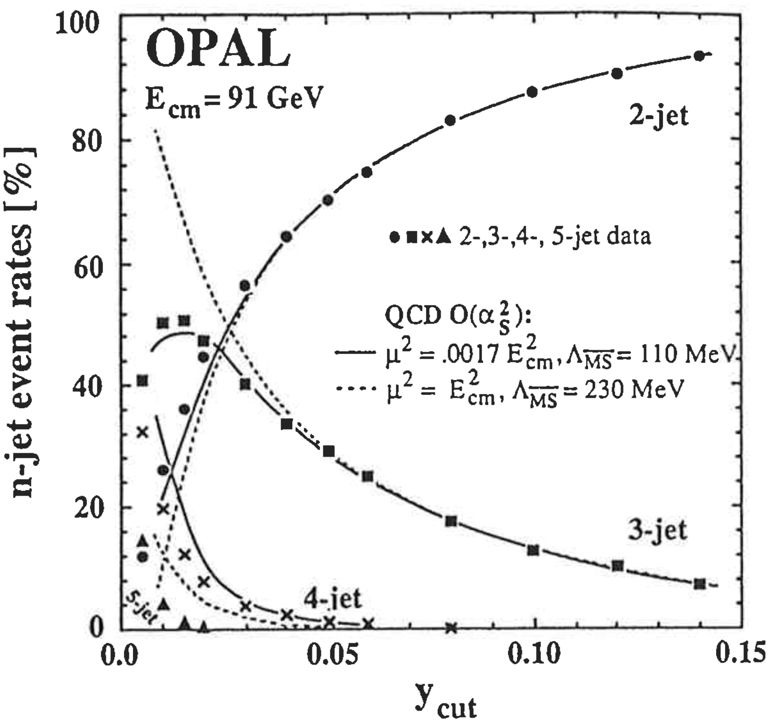


Fig. 3.11. Distribution in the Bengtsson-Zerwas angle at LEP. Figure from ref. [26].



.7. QCD fits to the jet rates at LEP, as measured by the OPAL collabo-
Figure from ref. [15].

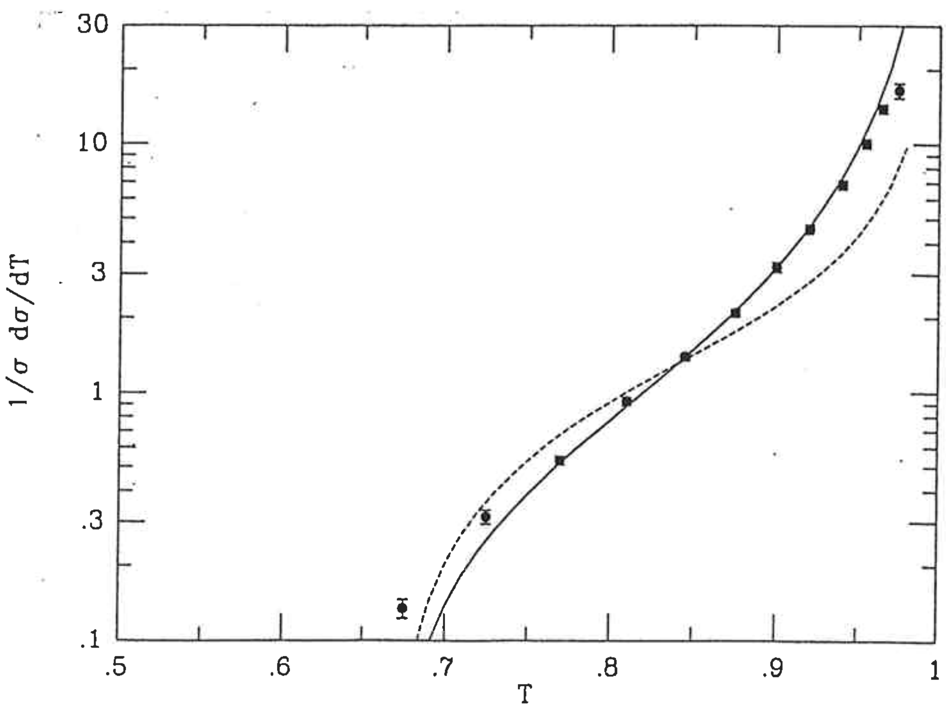
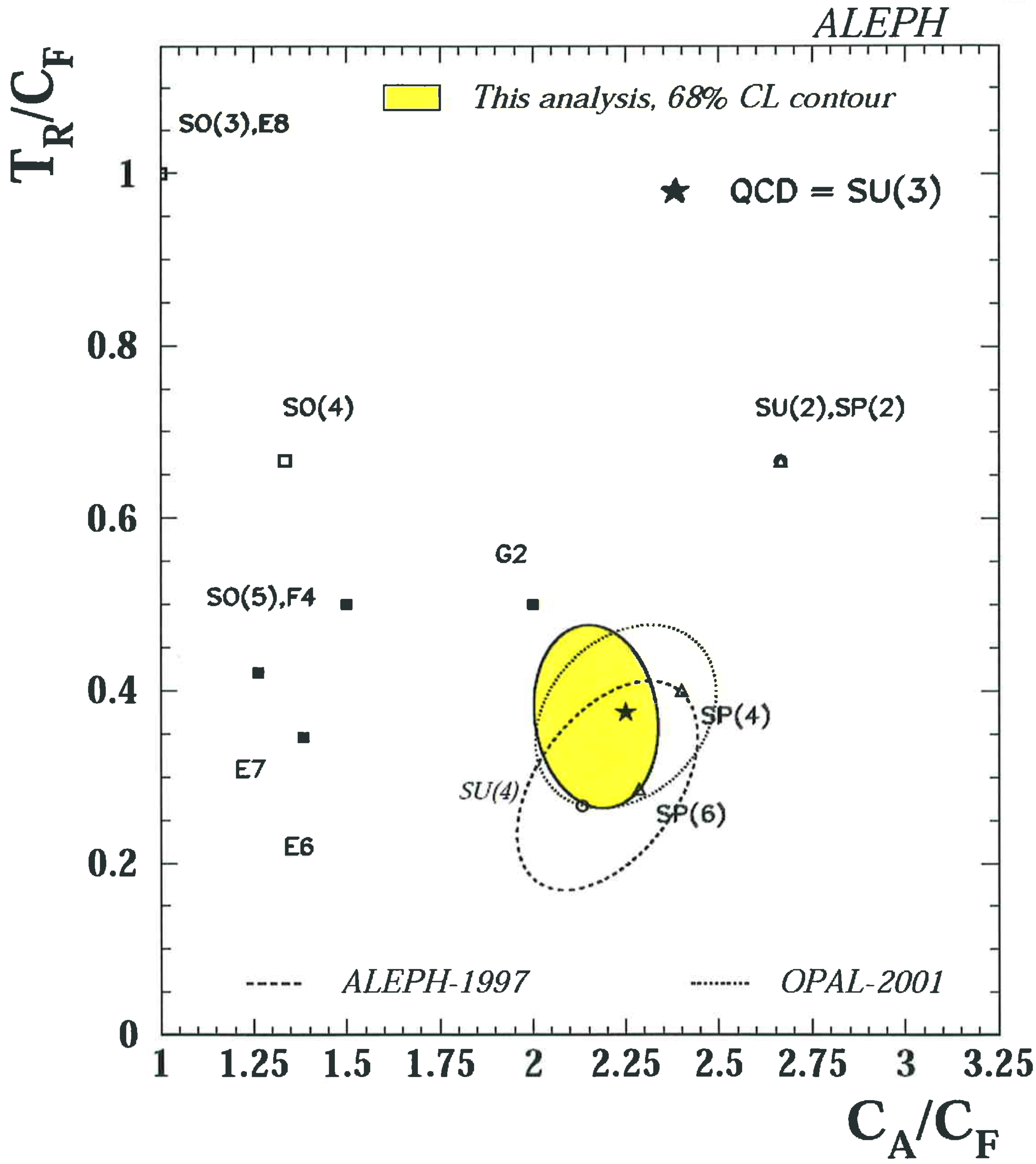


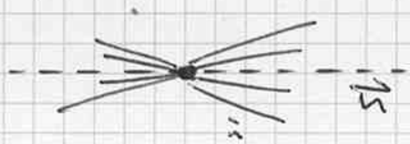
Fig. 3.9. The thrust distribution measured at LEP, showing data from the DELPHI collaboration [22] for $T < 0.98$, together with predictions of scalar gluon (dashed line) and vector gluon (solid line) theories.


ALEPH
This analysis, 68% CL contour
QCD = SU(3)
SO(3),E8
SO(4)
SU(2),SP(2)
G2
SO(5),F4
E7
E6
SU(4)
SP(4)
SP(6)
ALEPH-1997
OPAL-2001
T_R/C_F
C_A/C_F
0
0.2
0.4
0.6
0.8
1
1.25
1.5
1.75
2
2.25
2.5
2.75
3
3.25

• <u>Shape-Variablen:</u>

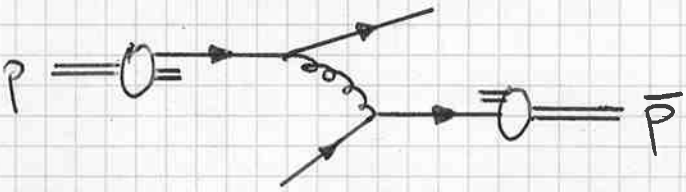
Thrust, Sphärizität, Massen, C-Parameter, ...

Thrust: $T = \max_{\vec{n}} \frac{\sum |\vec{p}_i \vec{n}|}{\sum |\vec{p}_i|}$ 3j: $\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{2\alpha_s}{3\pi}\left\{ \frac{2(3T^2-3T+2)}{T(1-T)} \log\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right\}$



2 jet: $T = 1$
sphär: $T = \frac{1}{2}$
$q\bar{q}g$: $T = \max x_i$

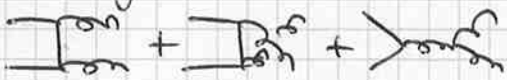
(c) <u>Jets in hochenergetischer $p\bar{p}$-Streuung bei hohem Transversalimpuls</u>



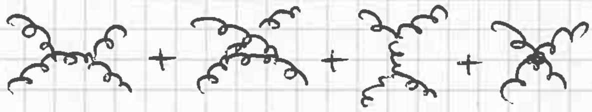
hohes $p_\perp$ im Subsystem $\Rightarrow$ kleine Raum-Zeitdistanz für Subprozeß-Ablauf

Rutherford-Prozeß: $q+q \to q+q$, $q+\bar{q} \to q+\bar{q}$ 

Compton-Streuung: $g+q \to g+q$, $g+\bar{q} \to g+\bar{q}$

Annihilation: $q+\bar{q} \to g+g$ 

Gluon-Fusion: $g+g \to q+\bar{q}$

Gluon-Streuung: $g+g \to g+g$ 

• Faltung mit Parton-Verteilungen:

$$\frac{d\sigma}{dA}(p\bar{p} \to j_1 j_2 + \dots) = \sum_{qg} \int_0^1 dx_1\, f_1(x_1, Q^2) \int_0^1 dx_2\, f_2(x_2, Q^2) \int d\hat{\sigma}(p_1 p_2 \to p_1' p_2', \hat{s} = x_1 x_2 s) \times \delta_1(A - A(p_i))$$

• Nachweis der <u>Rutherford-Streuung im Quark-Gluon-Sektor:</u>

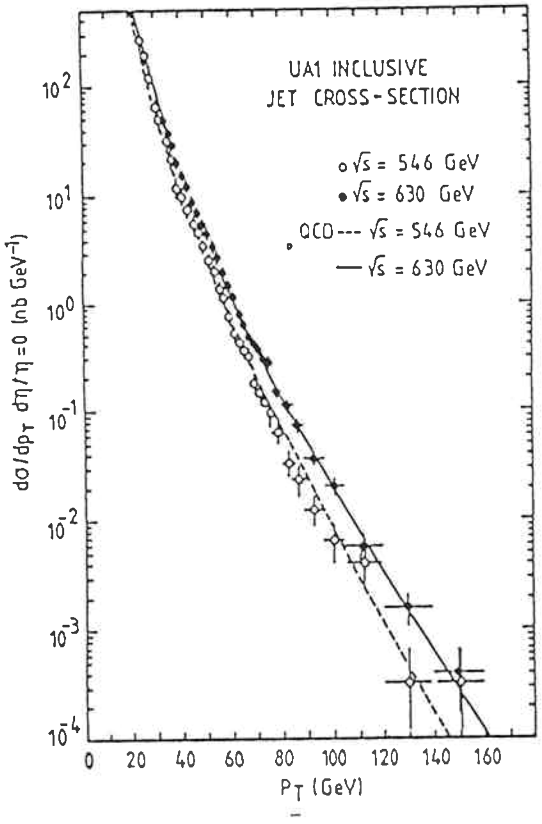
$$\frac{d\sigma^R}{d\cos\vartheta} \sim \frac{1}{\sin^4\frac{\vartheta}{2}} \sim \frac{1}{(1-\cos\vartheta)^2}$$ steil ansteigend für $\vartheta \to 0$

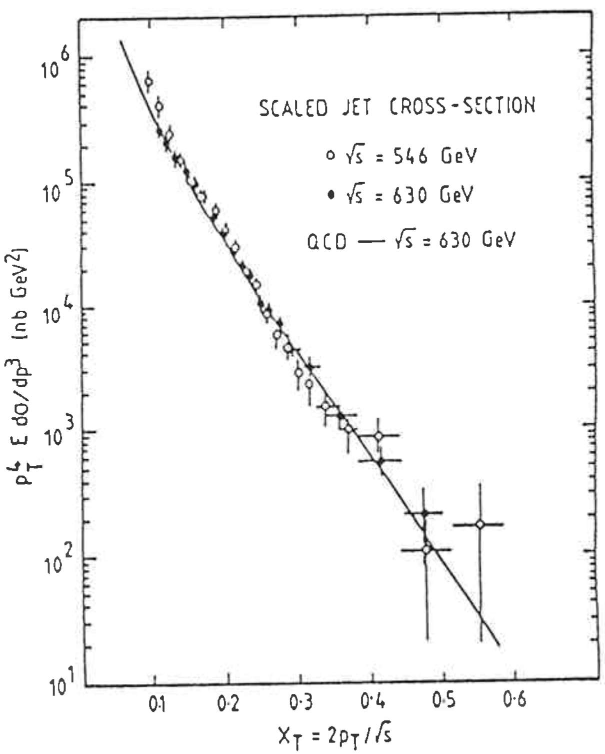
$$X = \frac{1+\cos\vartheta}{1-\cos\vartheta} \qquad dX \sim \frac{d\cos\vartheta}{(1-\cos\vartheta)^2} \Rightarrow \underline{\underline{\frac{d\sigma^R}{dX} = \text{flach}}}$$

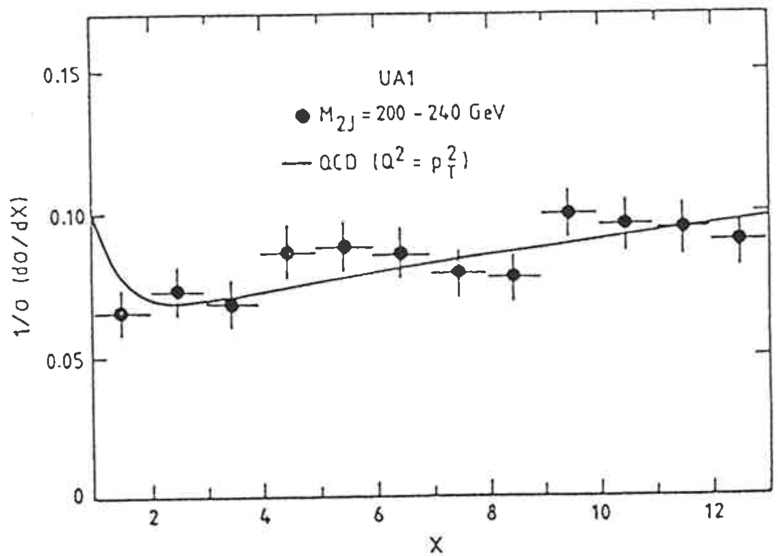
modulo: X-Abh. in $d\sigma^R$
Q^2-Abh. in $\alpha_s(Q^2)$, Quarkdichten

Tabelle 20-1 Die Querschnitte für die möglichen Parton-Parton-2-Teilchen-Reaktionen in führender Ordnung in α_s. Dabei numerieren wir mit i, j $(1 \leqslant i,\ j \leqslant f)$ die verschiedenen Quark-Flavors durch (s. Gl. (19-2)).

Reaktion	Streuquerschnitt $\frac{\hat{s}^2}{\pi \alpha_s^2} \frac{d\sigma}{d\hat{t}}$
$q^i + q^j \longrightarrow q^i + q^j \quad (i \neq j)$ $q^i + \bar{q}^j \longrightarrow q^i + \bar{q}^j \quad (i \neq j)$	$\frac{4}{9} \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$
$q^i + q^i \longrightarrow q^i + q^i$	$\frac{4}{9} \left(\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} + \frac{\hat{s}^2 + \hat{t}^2}{\hat{u}^2} \right) - \frac{8}{27} \frac{\hat{s}^2}{\hat{u}\hat{t}}$
$q^i + \bar{q}^i \longrightarrow q^j + \bar{q}^j \quad (i \neq j)$	$\frac{4}{9} \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2}$
$q^i + \bar{q}^i \longrightarrow q^i + \bar{q}^i$	$\frac{4}{9} \left(\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} + \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} \right) - \frac{8}{27} \frac{\hat{u}^2}{\hat{s}\hat{t}}$
$q^i + \bar{q}^i \longrightarrow G + G$	$\frac{32}{27} \frac{\hat{u}^2 + \hat{t}^2}{\hat{u}\hat{t}} - \frac{8}{3} \frac{\hat{u}^2 + \hat{t}^2}{\hat{s}^2}$
$G + G \longrightarrow q^i + \bar{q}^i$	$\frac{1}{6} \frac{\hat{u}^2 + \hat{t}^2}{\hat{u}\hat{t}} - \frac{3}{8} \frac{\hat{u}^2 + \hat{t}^2}{\hat{s}^2}$
$q^i + G \longrightarrow q^i + G$ $\bar{q}^i + G \longrightarrow \bar{q}^i + G$	$-\frac{4}{9} \frac{\hat{u}^2 + \hat{s}^2}{\hat{u}\hat{s}} + \frac{\hat{u}^2 + \hat{s}^2}{\hat{t}^2}$
$G + G \longrightarrow G + G$	$\frac{9}{2} \left(3 - \frac{\hat{u}\hat{t}}{\hat{s}^2} - \frac{\hat{u}\hat{s}}{\hat{t}^2} - \frac{\hat{s}\hat{t}}{\hat{u}^2} \right)$







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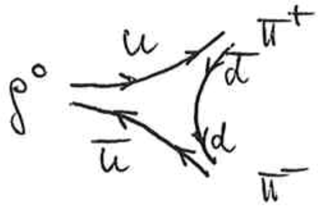
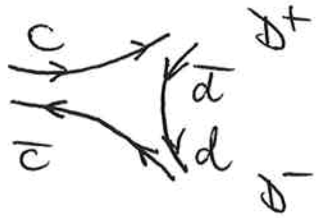
(d) Quarkonium-Zerfälle

$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$



$m_\rho > 2m_\pi$ Zweig-erlaubter Zerfall $\Rightarrow$ große Breite

$\psi = c\bar{c}$ (3097)
$\Upsilon = b\bar{b}$ (9460)



$m_\psi < 2m_D$ Zweig-erlaubter Zerfall nicht möglich

• leptonische Zerfälle: $\psi \to e^+e^-, \mu^+\mu^-$ [$q\bar{q}$]

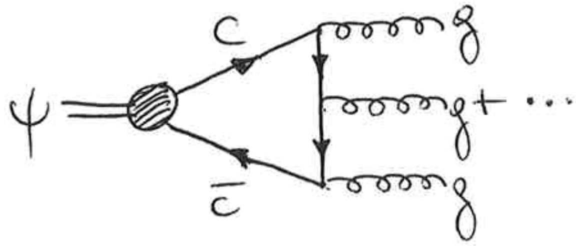
$$\Gamma(\psi \to l^+l^-) = \frac{16\pi\alpha^2}{m_\psi^2} Q_c^2 \, |\varphi(0)|^2 \qquad [\leftarrow \text{Positronium}]$$

$Q_c = +\frac{2}{3}$; $\varphi(0)$: Wellenfkt. am Ursprung [NR]

• hadronische Zerfälle: Quarkonia, für die Zweig-erlaubte Zerfälle nicht möglich sind, zerfallen in Gluonen
→ Jets bei hohen Energien [Υ, ...]

$1^{--} \not\to gg$: niedrigster Ortho-Kanal
[Yang]

$\psi = c\bar{c} \to 3g$
$\Upsilon = b\bar{b} \to 3g$



Annihilationsdistanz: $d \sim m_Q^{-1} \ll 1\,\text{fm} \Rightarrow$ asymptot. Freiheit

Techn.: $\Gamma(\psi \to ggg) = \sigma(c\bar{c} \to ggg) \times \left[v_R \, |\varphi_S(0)|^2 \right] \times \left[\frac{4}{3} \right]$

Spin-Mittelungs-Korrektur

$\Longrightarrow$ <u>Breite:</u> $\Gamma(Q\bar{Q} \to ggg) = \frac{160}{81}(\pi^2 - 9) \frac{\alpha_S^3(M^2)}{M^2} |\varphi_S(0)|^2$

$\psi = 0.05 \pm 0.01$ MeV

$\Upsilon = 0.04$ MeV

Quarkonia sind sehr schmale Resonanzen [$\leftarrow$ asympt. Freiheit]

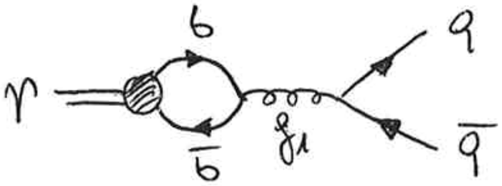
<u>Dalitz:</u> $\frac{1}{\Gamma} \frac{d\Gamma}{dx_1 dx_2} = \frac{6}{\pi^2 - 9} \frac{x_1^2(1-x_1)^2 + \dots}{x_1^2 x_2^2 x_3^2}$

<u>Jet-Energie:</u> $\frac{1}{\Gamma} \frac{d\Gamma}{dx_1} \approx 2x_1$

im Photon-Kanal $\Upsilon \to \gamma + gg$:

partielle Breite $\Gamma_\gamma / \Gamma_{tot} \sim \frac{\alpha \alpha_S^2}{\alpha_S^3} \sim \frac{\alpha}{\alpha_S}$ Meßmögl. kleines Λ

<u>Color-Ldg der Gluonen:</u> wäre g ein $U(1)$ Eichfeld, so wäre Zerfallsmodus: $\Upsilon \to g_{virt} \to q\bar{q}$



$\Longrightarrow$ 2 Jet-Endzustände = oft

nicht beobachtet $\Longrightarrow$ $\boxed{C[g] \neq 0}$

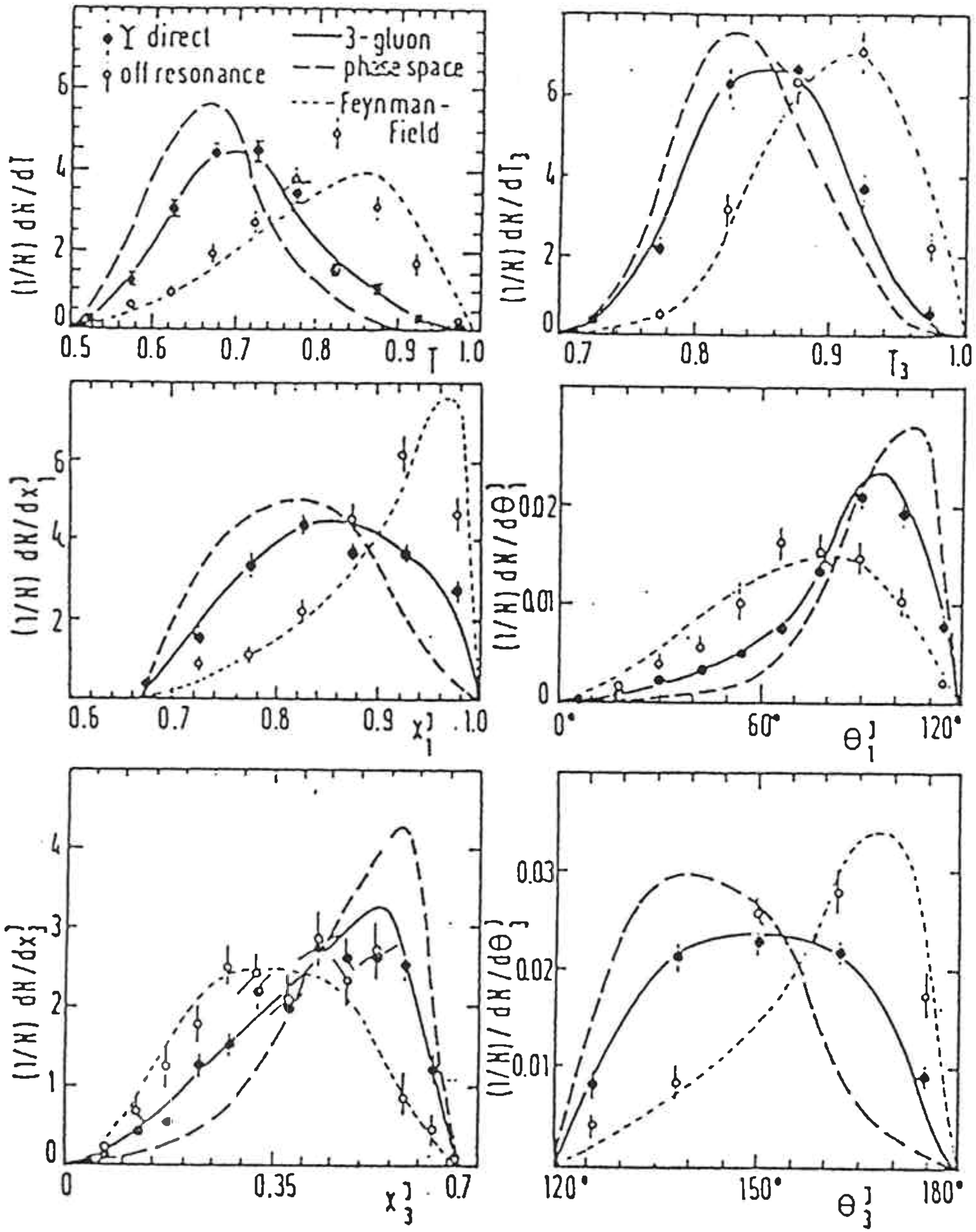
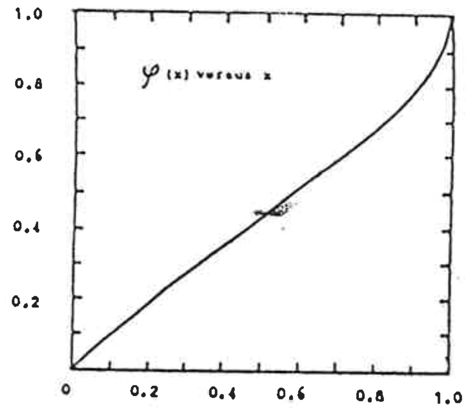
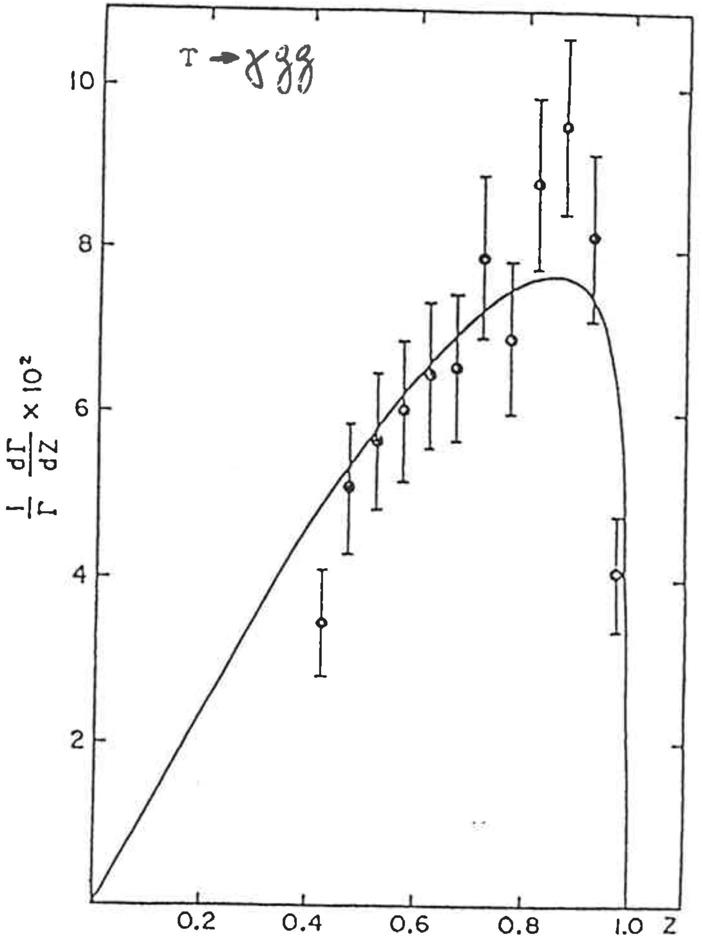


Fig. 2 Experimental distributions of thrust T, triplicity T_3, reconstructed gluon energies x_1^J, x_3^J and reconstructed angles θ_1^J, θ_3^J between gluons compared to Monte-Carlo calculations based on various models.





C. QCD BEI GROSSEN ABSTÄNDEN

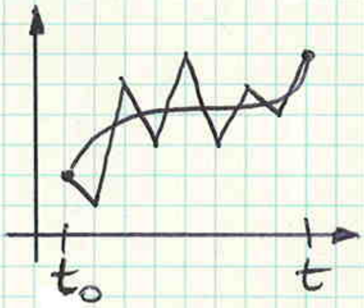
§1. Confinement-Potential

nicht-störungstheoretischer Bereich: Integration per Pfadintegrale

QM: $\langle x,t|x_0,t_0\rangle = \langle x| e^{-iH(t-t_0)}|x_0\rangle$

$$= \prod dx_i \langle x|e^{-iH\varepsilon}|x_n\rangle\langle x_n|\cdots|x_1\rangle\langle x_1|e^{-iH\varepsilon}|x_0\rangle$$

fT: $\langle x_2|e^{-iH\varepsilon}|x_1\rangle = \langle x_2|e^{-i\frac{P^2}{2m}\varepsilon}|x_1\rangle$



$$\sim \int dp\, \langle x_2|e^{-i\frac{p^2}{2m}\varepsilon}|p\rangle\langle p|x_1\rangle$$

$$\sim \int dp\, e^{-i\frac{p^2}{2m}\varepsilon + ip(x_2-x_1)}$$

$$\sim e^{i\frac{m}{2}\varepsilon\left(\frac{x_2-x_1}{\varepsilon}\right)^2} \sim e^{i\frac{m}{2}\varepsilon\dot{x}^2} \sim e^{i\varepsilon L} \qquad \dot{x}^2 = \left(\frac{x_2-x_1}{\varepsilon}\right)^2$$

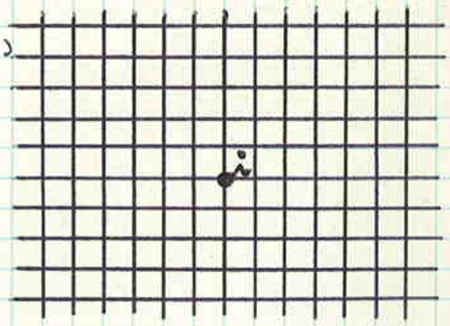
$$\langle x,t|x_0,t_0\rangle = \int_{Pf} \mathcal{D}x\, e^{iS}$$

W'amplitude = Summe über alle Geschichten mit Gewicht $\exp iS$

QFT: $\langle 0|T\{\varphi(x_2)\varphi(x_1)\}|0\rangle \sim \int \prod d\varphi_i\, \varphi_2\varphi_1\, e^{iS(\varphi)}$

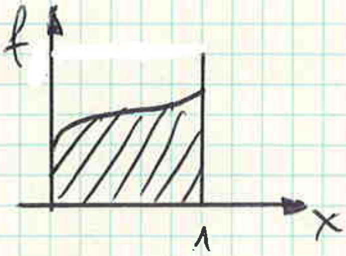
Lösung des Integrals durch Monte-Carlo-Methoden:

[nach Euklidisierung $t \to -ix_4$: $e^{iS} \to e^{-S}$]



1.) einfaches Integral: f

$$\int_0^1 dx\, f(x) = ?$$



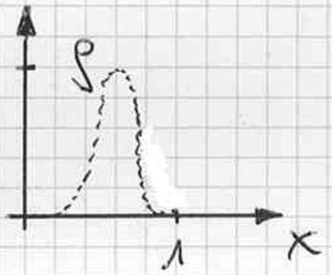
(i) wähle x_i zufällig verteilt; N Punkte

(ii) berechne $f(x_i) = f_i$

$$\longrightarrow \int_0^1 dx\, f(x) = \frac{1}{N}\sum_i f_i$$

2.) importance sampling:

$\int_0^1 dx\, \rho(x) f(x)$ ρ bedeutend nur in kleinem Teil des Phasenraums $I = \int_0^1 dx \left[\int_0^1 dy\, \Theta(\rho(x) - y) \right] f(x)$



(i) wähle x_i zufällig; berechne ρ_i;

(ii) wähle y_i zufällig; wenn $\rho_i > y_i$, dann berechne f_i, $i \in N_+$

$\longrightarrow I = \frac{1}{N} \sum_{i \in N_+} f_i$

3.) Hohe Dim.-zahl, Gewicht e^{-S_E}, einfache Observable:
stelle Sequenz von Konfigurationen $\{\varphi\}$ mit Verteilung e^{-S_E} ein; messe Observable Θ über Konfigurationen

$\langle \Theta \rangle = \sum_i \Theta_i / \sum_i \mathbb{1}$

<u>Theorem:</u> Startet man von einer beliebigen Konfiguration und ändert sie in konsekutiven Schritten mit bedingter Wahrscheinlichkeit $P(\mathcal{C}', \mathcal{C})$, so daß Glgw. $\Rightarrow$ Glgw.,

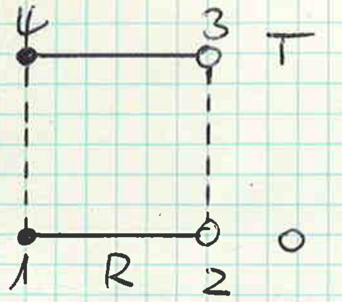
$e^{-S(\mathcal{C}')} = \sum_{\mathcal{C}} P(\mathcal{C}', \mathcal{C})\, e^{-S(\mathcal{C})}$, dann strebt das System automatisch ins Gleichgewicht.

<u>Metropolis:</u> $P(\mathcal{C}', \mathcal{C}) = \begin{cases} 1 & \text{für } S' < S \\ e^{-(S'-S)} & \text{für } S' > S \end{cases}$

WILSON Loop

Hantel-Operator: $h = \overline{Q}(y)\, e^{i g_s \int_x^y G^\mu ds_\mu}\, Q(x)$

[eichinvariant]



$\langle 0 | h^*(T)\, h(0) | 0 \rangle$

$= \sum_n \langle 0 | e^{HT} h^*(0) e^{-HT} | n \rangle \langle n | h(0) | 0 \rangle$

$= \sum_n |\langle 0 | h(0) | n \rangle|^2 e^{-E_n T}$

$\sim |\langle 0 | h(0) | \overline{Q}Q \rangle|^2 e^{-E_{min}(R) T}$ $\qquad E_{min}(R) = 2m + V(R)$

$= \langle 0 | \overline{Q}_4 P_{43} Q_3 \cdot \overline{Q}_2 P_{21} Q_1 | 0 \rangle$ $\qquad$ [Q_i = äußere Quellen]

$= \int \mathcal{D}G \; {}_Q\langle 0 | \overline{Q}_4 P_{43} Q_3 \cdot \overline{Q}_2 P_{21} Q_1 | 0 \rangle_Q \, e^{-S_G} + O\left(\frac{1}{m}\right)$

$\underset{3 \quad 2}{\sqcup}$ = Quarkpropagator im Hintergrundfeld G:

$$[\gamma^0 (i\partial_0 - g_s G_0) - m]\, S(x,y;G) = \delta_4(x-y)$$

$$\longrightarrow iS(x,y;G) = \Theta(x^0 - y^0)\, \delta_3(\vec{x}-\vec{y})\, e^{-im(x^0-y^0)}\, e^{-ig_s \int_y^x G^\mu ds_\mu} \propto P_{32}$$

$$\longrightarrow \boxed{e^{-V(R)T} \sim \int \mathcal{D}G \, \Pi P \, e^{-S}}$$

Integral gelöst mit MC Methoden

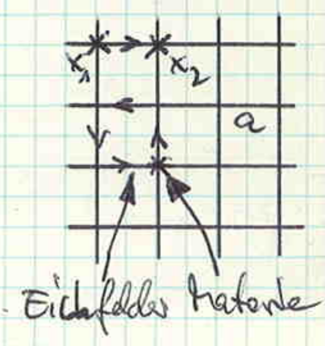
AUSWERTUNG der SU(3) ohne Quarkloops:

(i) Materiefelder definiert auf Gitterpunkten

(ii) Eichfelder definiert auf Links

Schwinger-Strom $j = \overline{\psi}(x_2)\, e^{ia \int_{x_1}^{x_2} dy_\mu G^\mu}\, \psi(x_1)$

[nicht-lokal, aber eichinvariant]



$G_\mu \longleftrightarrow U_\mu = e^{iaG_\mu} \longrightarrow e^{iaA_\mu}$ Link-Variable

kompakte Version suggestiv bei Bohm-Aharonov-Effekt

[$e^{i\oint dy^\mu A_\mu}$ = wesentliche Variable]

Wirkung:

$$S = -\frac{1}{2}\int d^4x\, \mathrm{Tr}\, G_{\mu\nu}^2 = -\frac{1}{2}\int d^4x\, \mathrm{Tr}\left\{\partial_\nu G_\mu - \partial_\mu G_\nu - i g_s [G_\mu, G_\nu]\right\}^2$$

$$\underset{\uparrow}{=} -\frac{1}{2 g_s^2}\int d^4x\, \mathrm{Tr}\left\{\partial_\nu G_\mu - \partial_\mu G_\nu - i [G_\mu, G_\nu]\right\}^2$$

$g_s G = \hat{G} \to G$

vergittert: Plaquette-Variable: $U_\Box = \prod U$

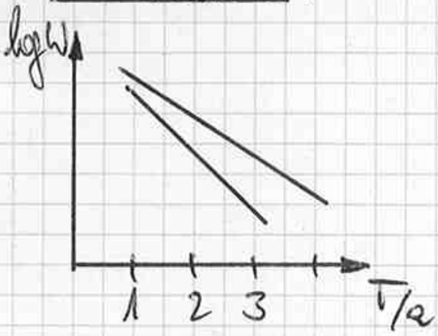
$$S = \frac{6}{g_s^2}\left\{1 - \frac{1}{6}\sum_P \mathrm{Tr}\left[U_\Box + U_\Box^+\right]\right\}$$

Entwicklung bis 2te Ordnung

→ Kontinuum-Wirkung

S gittereichinvariant

MESSUNG:



Steigung: V in Gittereinheiten

Fitansatz:

physikalische Einheiten: $V(R) = -\frac{\hat{\alpha}_s}{R} + \sigma R$

σ = Stringspannung

Gittereinheiten: $aV = -\frac{\hat{\alpha}_s}{R/a} + \sigma a^2 \frac{R}{a}$

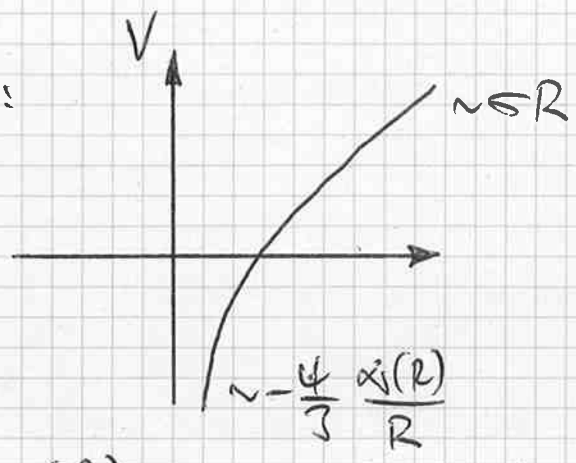
Fit: σa^2 gemessen im Gitter $\xrightarrow[a\,\varepsilon\,\text{amp}]{}$ $\underline{\underline{\boxed{\sqrt{\sigma}} \approx 400\text{ MeV}}}$ bestimmt

$\hat{\alpha}_s \approx 0.25$ gemessen

Meßresultate in qualitativer Übereinstimmung mit Erwartung F

Stringspannung:

(a) Richardson-Potential:



$$\left.\begin{array}{l}\vec{q}^{\,2}\text{ groß}\\ R\text{ klein}\end{array}\right\} \quad V(\vec{q}^{\,2}) \approx -\frac{4}{3}\frac{\alpha_s(\vec{q}^{\,2})}{\vec{q}^{\,2}} \qquad \alpha_s(\vec{q}^{\,2}) = \frac{12\pi}{(33-2N_F)\log\frac{\vec{q}^{\,2}}{\Lambda^2}}$$

$$V(R) \approx -\frac{4}{3}\frac{\alpha_s(R)}{R} \qquad \alpha_s(R) = \frac{12\pi}{(33-2N_F)\log\frac{1}{R^2\Lambda^2}}$$

$$\left.\begin{array}{l}R\text{ groß}\\ \vec{q}^{\,2}\text{ klein}\end{array}\right\} \quad \begin{array}{l}V(R) \approx \sigma R\\ V(\vec{q}^{\,2}) \approx -\frac{8\pi\sigma}{\vec{q}^{\,4}}\end{array}$$

Interpolation: Richardson-Potential

$$V(R) = -\frac{4}{3}\frac{48\pi^2}{33-2N_F}\int\frac{d^3\vec{q}}{(2\pi)^3}\,\frac{e^{i\vec{q}\vec{r}}}{\vec{q}^{\,2}\log\left[1+\frac{\vec{q}^{\,2}}{\Lambda^2}\right]}$$

$\longrightarrow \quad \sigma \approx \Lambda^2 \approx (400\ \text{MeV})^2$: Quarkonium-Spektroskopie

(b) Meson-String-Rotator:

$$dm = \gamma\sigma\, dl = \frac{\sigma\, dl}{\sqrt{1-\frac{v^2}{c^2}}} = L\frac{\sigma\, d\frac{l}{L}}{\sqrt{1-\left(\frac{l}{L}\right)^2}} = L\frac{\sigma\, dx}{\sqrt{1-x^2}}$$

Energiedichte — Masse, Energie

Gluon-Röhre $\left[\frac{v}{c} = \frac{l}{L}\right]$

$$dj = l\,dp = l\,v\,dm = L^2\frac{\sigma x^2\, dx}{\sqrt{1-x^2}} \quad \text{Drehimpuls}$$

$$\left.\begin{array}{l}m = \pi L\sigma\\ j = \frac{\pi}{2}L^2\sigma\end{array}\right\} \quad \boxed{j = \frac{m^2}{2\pi\sigma}}$$

Chew-Frautschi-Plot F

lin. Spin-Masse²-Beziehung

$\sigma = (420\ \text{MeV})^2$ — Rotationsinvariant F

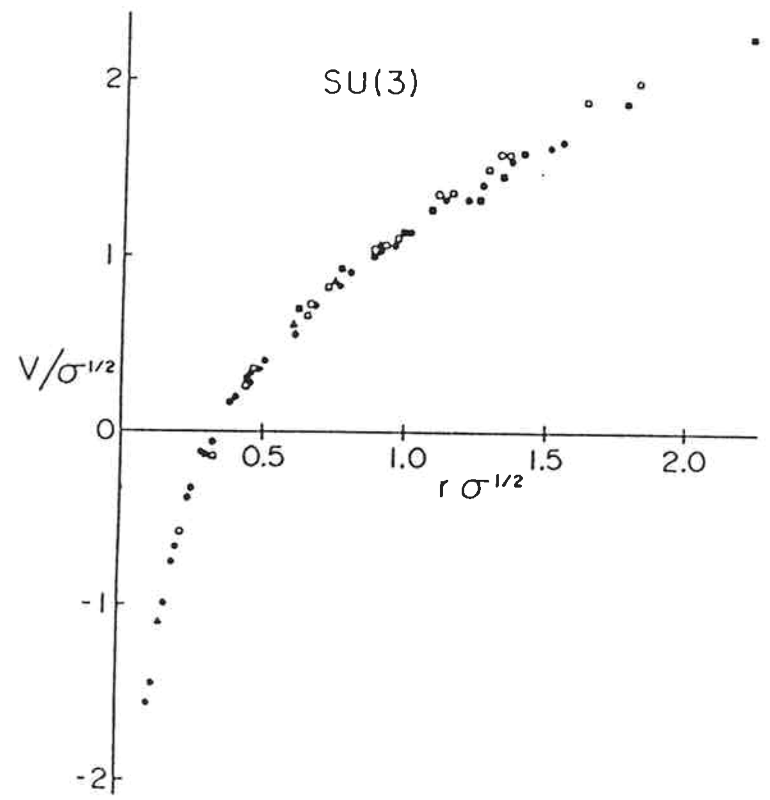
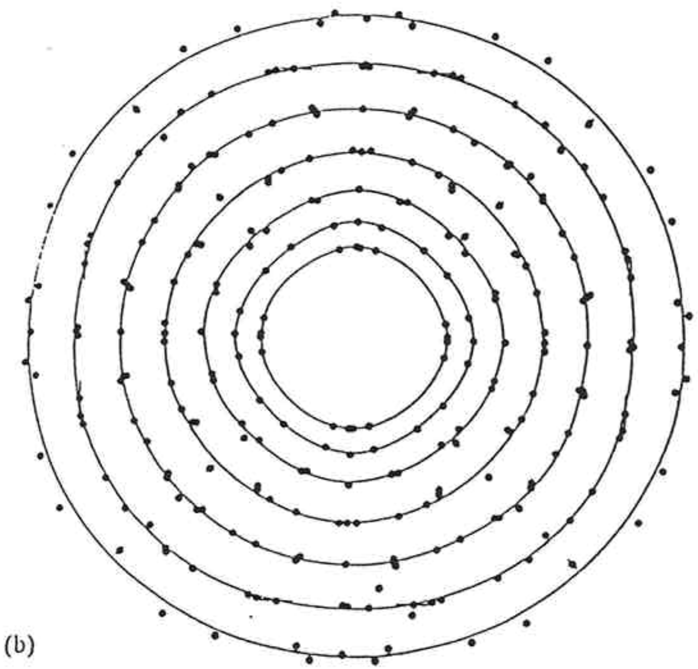


Figure 2: The q-$\bar{q}$ potential as obtained from different numerical simulations.



(b)

Fig. 1. Restoration of rotational invariance from (a) $\beta = 2$, $n_s = 8$, $n_t = 4$ to (b) $\beta = 2.25$, $n_s = 16$, $n_t = 6$; the equipotential curves are obtained through fits as described in the text.

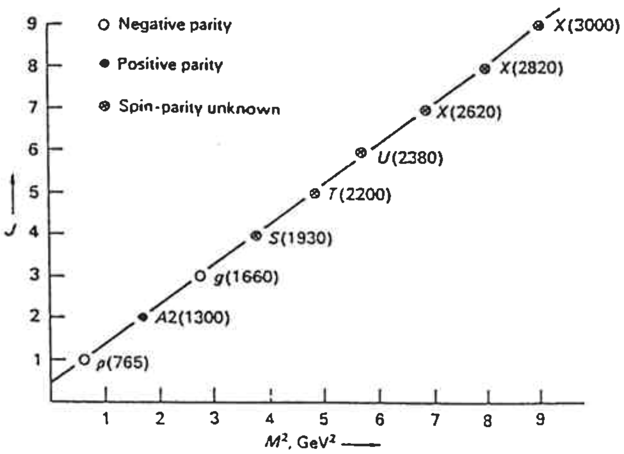


Fig. 7.34 Chew-Frautschi plot of nonstrange meson resonances, of $I = 1$ and spin, parity and G-parity, $J^{PG} = 1^{-+}, 2^{+-}, 3^{-+} \cdots$. The quantum numbers of only the first three states are known at present, the remainder having been plotted at the nearest integer spin value. The masses of the S, T, U, and X bosons are taken from Fig. 7.22.

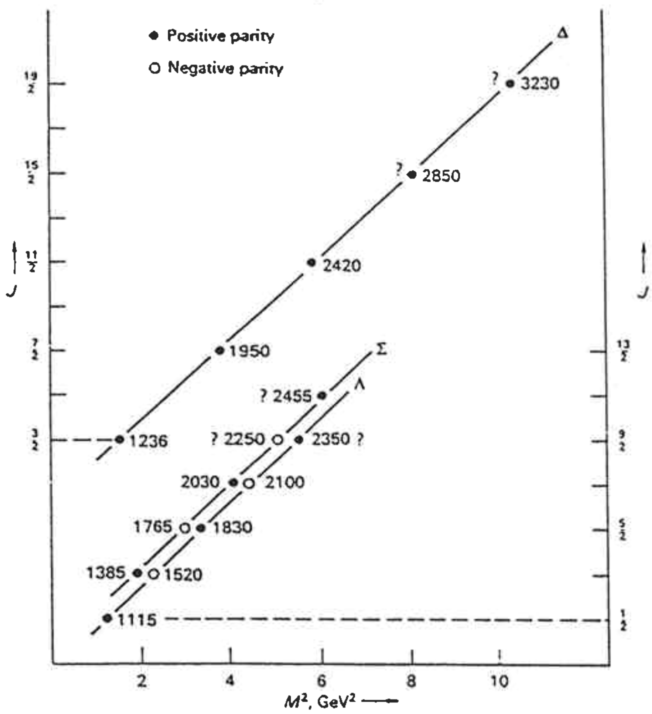


Fig. 7.33 Chew-Frautschi plots of fermion Regge trajectories. The trajectory marked Δ consists of the sequence $I = \frac{3}{2}$, $S = 0$, and $J^P = \frac{3}{2}^+, \frac{7}{2}^+, \frac{11}{2}^+ \cdots$; that marked Λ of the sequence $I = 0$, $S = -1$, $J^P = \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+ \cdots$; and that marked Σ of the sequence $I = 1$, $S = -1$, $J^P = \frac{3}{2}^+, \frac{5}{2}^-, \frac{7}{2}^+ \cdots$; resonances for which the spin-parity is not firmly established are indicated by a question mark.

§2. Chirale Invarianz

Operator-Transformation: $\phi_i' = e^{i\vec{\alpha}\vec{Q}}\phi e^{-i\vec{\alpha}\vec{Q}} = e^{i\vec{\alpha}\vec{T}}\phi$

Color-Operator SU(N)-Generator

Noether-Strom: $j_\mu^a = -\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_i)}\frac{\delta\phi_i}{\delta\alpha^a} - \frac{\partial\mathcal{L}}{\partial(\partial^\mu\bar{\phi}_i)}\frac{\delta\bar{\phi}_i}{\partial\alpha^a}$

$$\Longrightarrow \boxed{\partial^\mu j_\mu^a = -\frac{\delta\mathcal{L}}{\delta\alpha^a}} \quad \boxed{Q^a = \int d^3\vec{x}\, j_0^a(x)}$$

Lagrangedichte invariant: $\frac{\delta\mathcal{L}}{\delta\alpha^a} = 0 \Rightarrow \partial^\mu j_\mu^a = 0 \Rightarrow \dot{Q}^a = 0$

Quanten-Flavor-Dynamik: $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$

Massenmatrix: $M_{ij} = m_i\delta_{ij}$

$\mathcal{L} = \sum_i \bar{q}_i(i\partial\!\!\!/ - m_i)q_i = \bar{q}(i\partial\!\!\!/ - M)q$

(i) Vektorstrom: $q' = e^{i\vec{\alpha}\vec{T}}q \longrightarrow \bar{q}' = \bar{q}e^{-i\vec{\alpha}\vec{T}}$

$$\mathcal{L}' = \bar{q}\left[i\partial\!\!\!/ - e^{-i\vec{\alpha}\vec{T}}Me^{i\vec{\alpha}\vec{T}}\right]q \neq \mathcal{L}$$

$$\boxed{j_\mu^a = \bar{q}\gamma_\mu T^a q} \quad \boxed{Q^a = \int d^3\vec{x}\, q^+ T^a q}$$

Divergenz: $\partial^\mu j_\mu^a = -\frac{\delta\mathcal{L}}{\delta\alpha^a} = \bar{q}\left[M, iT^a\right]q = (m_j - m_k)\bar{q}_j iT^a_{jk}q_k$

(ii) Axialvektorstrom: $q' = e^{i\vec{\alpha}\vec{T}\gamma_5}q \Rightarrow \bar{q}' = \bar{q}e^{i\vec{\alpha}\vec{T}\gamma_5}$

$$\mathcal{L}' = \bar{q}\left[i\partial\!\!\!/ - e^{i\vec{\alpha}\vec{T}\gamma_5}Me^{i\vec{\alpha}\vec{T}\gamma_5}\right]q \neq \mathcal{L}$$

$$\boxed{j_{5\mu}^a = \bar{q}\gamma_\mu\gamma_5 T^a q} \quad \boxed{Q_5^a = \int d^3\vec{x}\, q^+\gamma_5 T^a q}$$

Divergenz: $\partial^\mu j^a_{5\mu} = \bar{q}\gamma_5\{M, iT^a\}q = (m_j+m_k)\bar{q}_j\gamma_5 iT^a_{jk}q_k$

- <u>Ladungsalgebra / Stromalgebra</u> für gleiche Zeiten $[SU_N \times SU_N]$:

$$\boxed{\begin{aligned}[Q^a(t), Q^b(t)] &= if_{abc}Q^c(t)\\ [Q^a_5(t), Q^b_5(t)] &= if_{abc}Q^c(t)\\ [Q^a(t), Q^b_5(t)] &= if_{abc}Q^c_5(t)\end{aligned}}$$

$Q^a_\pm = \frac{Q^a \pm Q^a_5}{2} = \int d^3x\, q^+ \frac{1\pm\gamma_5}{2}T^a q$

$$\boxed{\begin{aligned}[Q^a_\pm(t), Q^b_\pm(t)] &= if_{abc}Q^c_\pm(t)\\ [Q^a_+(t), Q^b_-(t)] &= 0\end{aligned}}$$

Unabhängig vom Invariant-Zustand erfüllen die Ladungen Q^a, Q^a_5 aufgrund der kanonischen Vertauschungsregeln die obige Lie'sche Algebra für gleiche Zeiten.

<u>VR</u>: $\{q^+_{i\alpha}(x), q_{i\beta}(y)\}_{ET} = \delta_{\alpha\beta}\delta_3(\vec{x}-\vec{y})$

$[T^a, T^b] = if_{abc}T^c$

- Stromerhaltung: (i) $m_j = m_k \Rightarrow \partial^\mu j^a_\mu = 0$ [i.a. $\partial^\mu j^a_{5\mu} \neq 0$]

(ii) $m_j = 0 \rightarrow \partial^\mu j^a_{5\mu} = \partial^\mu j^a_\mu = 0$ [Chiralität]

chirale Invariant: $M \rightarrow 0 \Rightarrow Q^a, Q^a_5$ erhalten

$\rightarrow$ Heisenberg-Multipletts: $q|0\rangle \rightarrow e^{i\vec{\alpha}\vec{Q}}q e^{-i\vec{\alpha}\vec{Q}}\underbrace{e^{i\vec{\alpha}\vec{Q}}|0\rangle}_{=|0\rangle} = e^{i\vec{\alpha}\vec{T}}q|0\rangle$

$\dot{\vec{Q}}_5 = i[H, \vec{Q}_5] = 0$

Parität: $P\vec{Q}P^{-1} = \vec{Q}$ $\qquad P\vec{Q}_5P^{-1} = -\vec{Q}_5$

$H|z\rangle = m_z|z\rangle$ $\qquad HQ_5|z\rangle = m_z Q_5|z\rangle \Rightarrow$ Entartung

$P|z\rangle = |z\rangle$ $\qquad PQ_5|z\rangle = -Q_5|z\rangle$

$\Rightarrow$ Paritätsdubletts

→ Nambu – Realisierung: chirale $SU_N \times SU_N$ spontan gebrochen

$$|0\rangle \to e^{i\vec{\alpha}\vec{T}\gamma_5}|0\rangle \quad \text{[Vakuum nicht invariant]}$$

- Kondensat:

(i) Heisenberg: $e^{i\vec{\alpha}\vec{Q}_5}|0\rangle = |0\rangle$

$$\langle 0|\bar{\psi}\psi|0\rangle = \langle 0| e^{-i\vec{\alpha}\vec{Q}_5} e^{i\vec{\alpha}\vec{Q}_5}\bar{\psi} e^{-i\vec{\alpha}\vec{Q}_5} e^{i\vec{\alpha}\vec{Q}_5}\psi e^{-i\vec{\alpha}\vec{Q}_5} e^{i\vec{\alpha}\vec{Q}_5}|0\rangle$$

$$= \left(e^{2i\vec{\alpha}\vec{T}\gamma_5}\right)_{ab} \langle 0|\bar{\psi}_a\psi_b|0\rangle$$

$$\Longrightarrow \langle 0|\bar{\psi}\psi|0\rangle = 0$$

(ii) Nambu: $\langle 0|\bar{\psi}\psi|0\rangle = \left(e^{2i\vec{\alpha}\vec{T}\gamma_5}\right)_{ab} \langle 0|\bar{\psi}_a\psi_b|0\rangle_{\alpha}$

$$\Longrightarrow \langle 0|\bar{\psi}\psi|0\rangle \neq 0 \text{ möglich}$$

<u>Goldstone-Theorem</u>:

N = Dimension der zur Symmetriegruppe gehörenden Algebra des vollständigen Lagrangedichte

M = Dimension der Algebra, die Vakuum invariant läßt nach Symmetriebrechung

⟹ Es gibt <u>N−M masselose Goldstonen in</u> der Theorie

<u>Bsp:</u> $SU_{2L} \times SU_{2R}$: $N = 6$ ⟶ SU_{2L+R}: $M = 3$

⟹ 3 Goldstonen: $\pi^{\pm}, \pi^0$ [u, d – Quarks] $\left[\frac{m_\pi^2}{m_\rho^2} \approx 2\%\right]$

$SU_{3L} \times SU_{3R}$: $N = 16$ ⟶ SU_{3L+R}: $M = 8$

⟹ 8 Goldstonen: $\pi^{\pm}, \pi^0, K^{\pm}, K^0, \bar{K}^0, \eta$

§3. PCAC – Hypothese

- Haag'sches Theorem:

  Sei ϕ ein Operator mit den Eigenschaften

  (i) Quantenzahlen korrekt

  (ii) $|\langle 0|\phi|1\rangle|^2 = 1$ [Normierung]

  Dann ist ϕ als Feldoperator verwendbar.

- Pion-Feld: $\langle 0| j^a_{5\mu}(x) | \pi^b(p)\rangle = i f_\pi p^\mu e^{-ipx}$

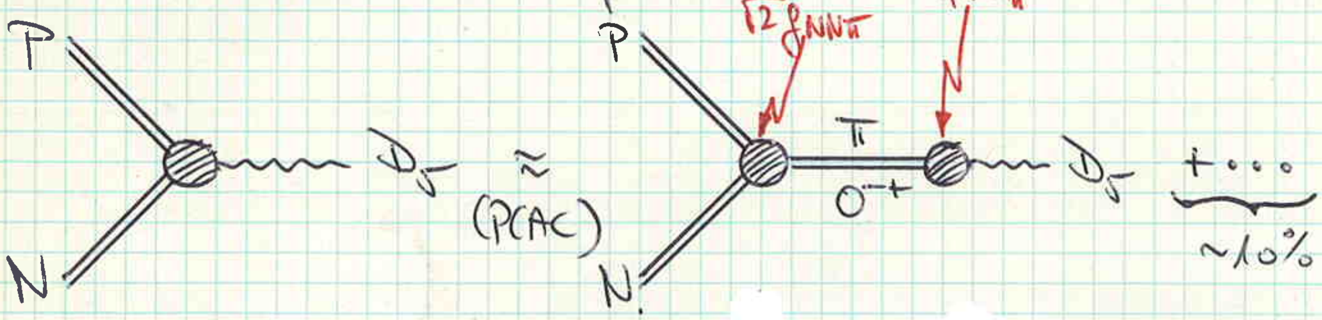
$$\Longrightarrow \langle 0| D^a_5(0) | \pi^b(p)\rangle = \langle 0|\partial^\mu j^a_{5\mu}(0)|\pi^b(p)\rangle = f_\pi m_\pi^2$$

$$\Longrightarrow \boxed{\phi^a_\pi(x) = \frac{D^a_5(x)}{f_\pi m_\pi^2}}$$

PCAC – Hypothese:

Wo auch immer ein Axialvektorstrom auftritt, kann man ihn durch ein 1-Pion-Feld ersetzen. [Pion-Pol-Dominanz]

Bsp.: (i) Nukleon-Formfaktoren

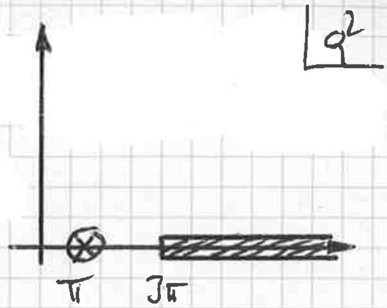


$$\langle P| j_{\mu 5}(0)|N\rangle = \bar{u}_P\left[\gamma_\mu\gamma_5 g_A(q^2) + q_\mu \gamma_5 g'_A(q^2)\right] u_N$$

$$\Longrightarrow \langle P|D_5(0)|N\rangle = i\bar{u}_P\left[-2m_N g_A(q^2) - q^2 g'_A(q^2)\right]\gamma_5 u_N$$

$$= i\bar{u}_P D(q^2)\gamma_5 u_N$$

$$D(q^2) = \frac{1}{\pi}\int_{\text{Cut}} \frac{\text{Im}\, D(q'^2)}{q'^2 - q^2}\, dq'^2$$



$$D(q^2) = \sqrt{2}\, g_{NN\pi} \frac{1}{q^2 - m_\pi^2} f_\pi m_\pi^2 \,(+\dots) \approx -2m_N g_A(q^2) - q^2 g_A'(q^2)$$

Da es kein masseloses Hadron gibt, kann g_A' bei $q^2=0$ keinen Pol haben.

$$\Longrightarrow \; -D(0) = \boxed{2m_N g_A(0) = \sqrt{2}\, g_{NN\pi}(0) f_\pi} \quad (\pm 10\%)$$

Goldberger – Treiman – Relation

(iii) QFT:

$$\delta_{ab} f_\pi m_\pi^2 = \langle 0| D_5^a(0) |\pi^b(p)\rangle \underset{LSZ}{=} i\int d^4x\, e^{-ipx}(\partial^2 + m_\pi^2)\langle 0|T\{D_5^a(0)\phi_\pi^b(x)\}|0\rangle$$

$$= i\lim_{p^2\to m_\pi^2}(m_\pi^2 - p^2)\int d^4x\, e^{-ipx}\langle 0|T\{D_5^a(0)\phi_\pi^b(x)\}|0\rangle$$

$$= i\lim_{p^2\to m_\pi^2} \frac{m_\pi^2-p^2}{f_\pi m_\pi^2}\int d^4x\, e^{-ipx} \langle 0|T\{D_5^a(0)\,\partial^\mu j_{5\mu}^b(x)\}|0\rangle$$

$$\partial^\mu T\{D_5^a(0)\, j_{5\mu}^b(x)\} = T\{D_5^a(0)\,\partial^\mu j_{5\mu}^b(x)\} + \delta(x^0)\left[j_{50}^b(x) D_5^a(0) - D_5^a(0)\, j_{50}^b(x)\right]$$

$$\Longrightarrow \delta_{ab} f_\pi m_\pi^2 = i\lim_{p^2\to m_\pi^2} \frac{m_\pi^2-p^2}{f_\pi m_\pi^2}(ip^\mu)\int d^4x\, e^{-ipx}\langle 0|T\{D_5^a(0)\, j_{5\mu}^b(x)\}|0\rangle$$

$$+ i\lim_{p^2\to m_\pi^2} \frac{m_\pi^2-p^2}{f_\pi m_\pi^2}\int d^4x\, e^{i\vec p\vec x}\,\delta(x^0)\langle 0|\left[D_5^a(0), j_{50}^b(x)\right]|0\rangle$$

PCAC: $\lim\limits_{p^2\to m_\pi^2} \approx \lim\limits_{p_\mu\to 0}$ (bis auf ~10%) $\quad \left[(p^2-m_\pi^2)\frac{F(m_\pi^2)}{p^2-m_\pi^2}\approx F(0)\right]$

$$\Longrightarrow \delta_{ab} f_\pi m_\pi^2 = \frac{-i}{f_\pi}\langle 0|\left[Q_5^b, D_5^a(0)\right]|0\rangle$$

mit $Q_5^b = \int_{x^0=0} d^3\vec x\; j_{50}^b(x)$

$$\partial_5^a = \partial^\mu j^a_{5\mu} = (m_j+m_k)\,\bar q_j\, i\gamma_5 T^a_{jk}\, q_k = \bar q\, i\gamma_5 \{M, T^a\}\, q$$

$$Q_5^b = \int_{x^0=0} d^3\vec x\; q^\dagger \gamma_5 T^b q \qquad\qquad M_{ij} = m_i\,\delta_{ij}$$

VR: $[Q_5^b, \partial_5^a(0)] = -i\bar q\,\{T^b,\{T^a,M\}\}\,q$

$$\Longrightarrow \delta_{ab}\, f_\pi^2 m_\pi^2 = -\langle 0|\bar q\{T^b,\{T^a,M\}\}q|0\rangle$$

Kondensate: $\langle 0|\bar q_a q_b|0\rangle = \delta_{ab}\,\langle 0|\bar q_a q_a|0\rangle$

$$\Longrightarrow f_\pi^2 m_\pi^2 = -\langle 0|\bar u u|0\rangle\, \mathrm{Tr}\{T^b,\{T^a,M\}\}$$

$$m_j \approx m_k \Longrightarrow \mathrm{Tr}\{T^b,\{T^a,M\}\} \approx \delta_{ab}\sum_k m_k$$

$$\boxed{f_\pi^2 m_\pi^2 = -\sum_k m_k \langle \bar u u\rangle_0}$$ Gell-Mann / Oakes / Renner

f_π aus Zerfall $\pi^+ \to \mu^+\nu_\mu$ bestimmt: $f_\pi = 94$ MeV

$\Longrightarrow$ Bestimmung der Quarkmassen + -kondensate

Pion erhält Masse durch Quarkmassen

(iii) Verallgemeinerung:

$$\langle z_1|O(0)|z_2,\pi^a(p)\rangle \overset{LSZ}{\underset{PCAC}{=}} i\int d^4x\, e^{-ipx}(\partial^2+m_\pi^2)\langle z_1|T\{O(0)\,\phi_\pi^a(x)\}|z_2\rangle$$

$$= i\lim_{p^2\to m_\pi^2}(m_\pi^2-p^2)\int d^4x\, e^{-ipx}\langle z_1|T\Big\{O(0)\,\frac{\partial^\mu j^a_{5\mu}(x)}{f_\pi m_\pi^2}\Big\}|z_2\rangle$$

$$T\{A(0)\,\partial_\mu B^\mu(x)\} = \partial_\mu T\{A(0)\,B^\mu(x)\} + \delta(x^0)\,[A(0), B^0(x)]$$

$$\Longrightarrow \langle z_1|O(0)|z_2,\pi^a(p)\rangle = i\lim_{p^2\to m_\pi^2}\frac{m_\pi^2-p^2}{f_\pi m_\pi^2}\Big\{ ip_\mu\int d^4x\, e^{-ipx}\langle z_1|T\{O(0)\,j^{a\mu}_5(x)\}|z_2\rangle + \int d^4x\, e^{i\vec p\vec x}\,\delta(x^0)\langle z_1|[O(0), j^a_{50}(x)]|z_2\rangle$$

$$\lim_{p^2 \to m_\pi^2} \approx \lim_{p \to 0}$$

$$\boxed{\begin{aligned} &\langle \alpha_1 | \mathcal{O}(0) | \alpha_2, \pi^a(p) \rangle = \frac{i}{f_\pi} \langle \alpha_1 | \left[\mathcal{O}(0), Q_5^a \right] | \alpha_2 \rangle \\ &\text{mit } Q_5^a = \int_{x^0=0} d^3\vec{x}\, j_{50}^a(x) \end{aligned}}$$

$\Longrightarrow$ Soft Pion Theorems

§4. Goldstone-Theorem [Nuovo Cimento 19 (1961) 15]

N = Dimension der zur Symmetriegruppe gehörenden Algebra der vollständigen Lagrangedichte

M = Dimension der Algebra, die Vakuum invariant läßt nach Symmetriebrechung

$\Longrightarrow$ Es gibt <u>$N-M$ masselose Goldstonen</u> in der Theorie.

<u>Beweis</u>: $\mathcal{L}(\varphi, \partial\varphi)$ invariant unter Symmetriegruppe

Noether-Ströme erhalten: $\partial_\mu V^\mu(x) = 0$

erhaltene Ladung: $Q = \int\limits_{t=0} d^3\vec{x}\, V^0(0,\vec{r})$

$$\Longrightarrow \quad \boxed{[Q, \varphi] = T\varphi}$$

Symmetrie gebrochen: $\langle 0|\varphi|0\rangle = \langle\varphi\rangle = v \neq 0$

$v_\mu(k) = \int d^4x\, e^{ikx} \langle 0|[V_\mu(x), \varphi]|0\rangle$ erfüllt:

Symmetriebedingung: $k_\mu v^\mu(k) = 0$

Symmetrie gebrochen:

$$\int dk^0\, v^0(k^0, \vec{0}) = \int dx^0 \int dk^0\, e^{ik^0x^0} \int d^3\vec{x}\, \langle 0|[V^0(t,\vec{x}), \varphi(0)]|0\rangle$$

$$= 2\pi \langle 0|[Q, \varphi]|0\rangle = 2\pi T v \neq 0$$

$a = 1, \ldots, M: \; T^a v = 0$

$a = M+1, \ldots, N: \; T^a v \neq 0 \quad \Longrightarrow N-M$

Spektraldarstellung:

$$v_\mu(k) = (2\pi)^4 \sum_n \Big\{ \langle 0|V_\mu|n\rangle \langle n|\varphi|0\rangle \delta_4(k-p_n)\, \Theta(k^0) - \langle 0|\varphi|n\rangle \langle n|V_\mu|0\rangle \delta_4(k+p_n)\, \Theta(-k^0) \Big\}$$

$\longrightarrow$ Lorentzinvariant: $\langle 0|V_\mu|n\rangle = f_n p_n^\mu$

$\longrightarrow$ positives Energiespektrum: 1. Summe: $\Theta(k^0) = \frac{1}{2}[1+\varepsilon(k^0)]$

2. Summe: $\Theta(-k^0) = \frac{1}{2}[1-\varepsilon(k^0)]$

$$v_\mu(k) = k_\mu \left[\sigma_+(k^2) + \varepsilon(k^0)\sigma_-(k^2)\right]$$

$$\sigma_\pm(k^2) = \frac{1}{2}(2\pi)^4 \sum_n \Big\{ \langle n|\varphi|0\rangle f_n \delta_4(k-p_n) \pm \langle 0|\varphi|n\rangle f_n^* \delta_4(k+p_n) \Big\}$$

$$k_\mu v^\mu(k) = 0 \implies \boxed{k^2 \sigma_\pm(k^2) = 0}$$

Lösung: $\sigma_\pm(k^2) = s_\pm \delta(k^2)$

$$v_\mu(k) = k_\mu \left[s_+ + s_- \varepsilon(k^0)\right] \delta_1(k^2)$$

$$\int_{-\infty}^{\infty} dk^0\, v_0(k^0, \vec{0}) = s_- 2 \int_0^\infty dk^0\, k^0 \delta_1(k^2) = s_- \neq 0$$

$$\implies \boxed{v_\mu(k) = \left[s_+ + s_- \varepsilon(k^0)\right] \delta_1(k^2) k_\mu \quad \text{mit } s_- \neq 0}$$

$\hookrightarrow$ Es gibt Zustände mit $p_n^2 = 0 \implies \boxed{m=0}$

Goldstonen

q.e.d.