Aufgabe 1

1

$$F(q^2) = \int d^3\vec{r}\, e^{-i\vec{q}\vec{r}} \rho(r)$$

$$\rho(r) = \int \frac{d^3\vec{q}}{(2\pi)^3} F(q^2) e^{i\vec{q}\vec{r}}$$

$$F(q^2) = \int r^2 dr \int_{-1}^{1} dt\, 2\pi\, e^{-iqrt} \rho(r)$$

$$= 2\pi \int r^2 dr \frac{e^{iqr} - e^{-iqr}}{iqr} \rho(r)$$

$$\boxed{F(q^2) = 4\pi \int_0^\infty r^2 dr \frac{\sin qr}{qr} \rho(r)}$$

$$\boxed{\rho(r) = \frac{1}{2\pi^2} \int_0^\infty q^2 dq \frac{\sin qr}{qr} F(q^2)}$$

$$F(q^2) = 4\pi \int_0^\infty r^2 dr \left[1 - \frac{q^2 r^2}{6}\right] \rho(r) + O(q^4)$$

$$= F(0) - \frac{q^2}{6} F(0) \langle r^2 \rangle + O(q^4)$$

$$= F(0) \left[1 - \frac{q^2 \langle r^2 \rangle}{6}\right] + O(q^4)$$

$$\rho(r) = \frac{1}{2\pi^2}\int_0^\infty q^2 dq \, \frac{\sin qr}{qr} \, \frac{1}{\left(1+\frac{q^2}{M^2}\right)^2}$$

$$= \frac{M^2}{2\pi^2 r}\int_0^\infty dx \, \frac{x \sin Mrx}{(1+x^2)^2}$$

$$= \frac{M^2}{4\pi^2 r}\left\{ \cancel{- \frac{\sin Mrx}{1+x^2}\Big|_0^\infty} \xrightarrow{} 0 + Mr\int_0^\infty dx \, \frac{\cos Mrx}{1+x^2} \right]$$

$$= \frac{M^3}{4\pi^2}\underbrace{\int_0^\infty dx \frac{\cos Mrx}{1+x^2}}_{= \frac{\pi}{2} e^{-Mr}} = \frac{M^3}{8\pi} e^{-Mr}$$

$$\boxed{\rho(r) = \frac{M^3}{8\pi} e^{-Mr}}$$

$$F(q^2) = 1 - 2\frac{q^2}{M^2} + O(q^4) = 1 - \frac{q^2}{6}\langle r^2 \rangle + O(q^4)$$

$$\Rightarrow \boxed{\langle r^2 \rangle = \frac{12}{M^2}}$$

$$\langle r^2 \rangle = \frac{\int_0^\infty r^4 dr\, e^{-\Lambda r}}{\int_0^\infty r^2 dr\, e^{-\Lambda r}}$$

$$\int_0^\infty r^4 dr\, e^{-\Lambda r} = \cancel{-\frac{1}{\Lambda} r^4 e^{-\Lambda r}\Big|_0^\infty}^{\to 0} + \frac{4}{\Lambda}\int_0^\infty r^3 dr\, e^{-\Lambda r}$$

$$= \cancel{-\frac{4}{\Lambda^2} r^3 e^{-\Lambda r}\Big|_0^\infty}^{\to 0} + \frac{12}{\Lambda^2}\int_0^\infty r^2 dr\, e^{-\Lambda r}$$

$$\boxed{\langle r^2 \rangle = \frac{12}{\Lambda^2}}$$

Aufgabe 2

$$F(q^2) = \int d^3\vec{r}\, e^{-i\vec{q}\vec{r}} \rho(\vec{r})$$

$$\rho(\vec{r}) = \delta_3(\vec{r}) - c\,\delta_1(r - r_0)$$

$$Q = \int d^3\vec{r}\, \rho(\vec{r}) = F(0) \overset{!}{=} 0$$

$$F(q^2) = 1 - 2\pi c \int_0^\infty r^2 dr \int_{-1}^{1} dt\, \delta(r - r_0)\, e^{-iqrt}$$

$$= 1 - 2\pi c\, r_0^2 \frac{e^{iqr_0} - e^{-iqr_0}}{iqr_0} = 1 - c4\pi \frac{r_0}{q} \sin q r_0$$

$$F(0) = 1 - 4\pi\, r_0^2 C = 0$$

$$C = \frac{1}{4\pi r_0^2}$$

$$\boxed{F(q^2) = 1 - \frac{\sin q r_0}{q r_0}}$$

$$\frac{d\sigma}{d\Omega} = \left(1 - \frac{\sin q r_0}{q r_0}\right)^2 \left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth}}$$