Musterlösung

Aufgabe 1: $\mathcal{L} = \bar{q}\,(i\not\partial - m)\,q$

a) $q' = e^{i\alpha^a T^a} q \Longrightarrow \frac{\delta q'}{\delta \alpha^a} = i T^a e^{i\alpha^a T^a} q = i T^a q'$

$$j^a_\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_j)} \frac{\delta q_j}{\delta \alpha^a} = -\bar{q}\, i\gamma_\mu\, i T^a q = \bar{q}\,\gamma_\mu T^a q$$

$$Q^a = \int d^3\vec{x}\, j^a_0(x) = \int d^3\vec{x}\, \underbrace{\bar{q}(x)}_{q^+(x)\gamma_0}\gamma_0 T^a q(x) = \int d^3\vec{x}\, q^+(x) T^a q(x)$$

b) $q' = e^{i\alpha^a T^a \gamma_5} q \Longrightarrow \frac{\delta q'}{\delta \alpha^a} = i T^a \gamma_5 q'$

$$j^a_{5\mu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_j)} \frac{\delta q_j}{\delta \alpha^a} = -\bar{q}\, i\gamma_\mu\, i T^a \gamma_5 q = \bar{q}\,\gamma_\mu \gamma_5 T^a q$$

$$Q^a_5 = \int d^3\vec{x}\, j^a_{50}(x) = \int d^3\vec{x}\, q^+(x)\gamma_5 T^a q(x)$$

Aufgabe 2:

$$[T^a, T^b] = i f_{abc} T^c$$

$$\{q^+_{i\alpha}(x), q_{j\beta}(y)\}_{x^0=y^0} = \delta_{ij}\delta_{\alpha\beta}\delta_3(\vec{x}-\vec{y})$$

$$\{q^+_{i\alpha}(x), q^+_{j\beta}(y)\}_{x^0=y^0} = \{q_{i\alpha}(x), q_{j\beta}(x)\} = 0$$

$A, B \in \{\mathbb{1}, \gamma_5\}$:

$$[Q^a_A, Q^b_B]_{ET} = \int d^3\vec{x}\, d^3\vec{y}\, [q^+(x) A T^a q(x), q^+(y) B T^b q(y)]_{x^0=y^0}$$

$$= A_{\alpha\beta} B_{\gamma\delta} T^a_{ij} T^b_{ke} \int d^3\vec{x}\, d^3\vec{y}\, [q^+_{i\alpha}(x) q_{j\beta}(x), q^+_{k\gamma}(y) q_{e\delta}(y)]_{x^0=y^0}$$

$$= A_{\alpha\beta} B_{\gamma\delta} T^a_{ij} T^b_{ke} \int d^3\vec{x}\, d^3\vec{y}\, \Big\{ q^+_{i\alpha}(x) \{q_{j\beta}(x), q^+_{k\gamma}(y)\}_{x^0=y^0} q_{e\delta}(y)$$
$$- q^+_{k\gamma}(y) \{q^+_{i\alpha}(x), q_{e\delta}(y)\}_{x^0=y^0} q_{j\beta}(y) \Big\}$$

$$= A_{\alpha\beta} B_{\gamma\delta} T^a_{ij} T^b_{ke} \int_{x^0=y^0} d^3\vec{x}\, d^3\vec{y} \Big\{ q^+_{i\alpha}(x) q_{e\delta}(y) \delta_{jk} \delta_{\beta\gamma} \delta_3(\vec{x}-\vec{y})$$
$$- q^+_{k\gamma}(y) q_{j\beta}(x) \delta_{ie} \delta_{\alpha\delta} \delta_3(\vec{x}-\vec{y}) \Big\}$$

$$= \int d^3\vec{x} \Big\{ q^+(x) AB T^a T^b q(x) - q^+(x) BA T^b T^a q(x) \Big\}$$

$A = B = 1\!\!1$: $[Q^a, Q^b] = \int d^3\vec{x}\, q^+(x) [T^a, T^b] q(x) = i f_{abc} Q^c$

$A = B = \gamma_5$: $\gamma_5^2 = 1\!\!1 \Rightarrow [Q^a_5, Q^b_5] = \int d^3\vec{x}\, q^+(x) [T^a, T^b] q(x) = i f_{abc} Q^c$

$A = 1, B = \gamma_5$: $[Q^a, Q^b_5] = \int d^3\vec{x}\, q^+(x) \gamma_5 [T^a, T^b] q(x) = i f_{abc} Q^c_5$

$$[Q^a_\pm, Q^b_\pm] = \frac{1}{4} [Q^a \pm Q^a_5, Q^b \pm Q^b_5]$$
$$= \frac{1}{4} \Big\{ [Q^a, Q^b] + [Q^a_5, Q^b_5] \pm [Q^a, Q^b_5] \pm [Q^a_5, Q^b] \Big\}$$
$$= \frac{i}{2} f_{abc} \Big\{ Q^c \pm \frac{1}{2} Q^c_5 \mp \frac{(-1)}{2} Q^c_5 \Big\} \quad (\text{wegen } f_{bac} = -f_{abc})$$
$$= i f_{abc} \frac{Q^c \pm Q^c_5}{2} = i f_{abc} Q^c_\pm$$

$$[Q^a_+, Q^b_-] = \frac{1}{4} [Q^a + Q^a_5, Q^b - Q^b_5]$$
$$= \frac{1}{4} \Big\{ [Q^a, Q^b] - [Q^a_5, Q^b_5] + [Q^a_5, Q^b] - [Q^a, Q^b_5] \Big\}$$
$$= \frac{i}{4} f_{abc} \Big\{ Q^c - Q^c \underset{\substack{\uparrow \\ f_{bac} = -f_{abc}}}{+} Q^c_5 - Q^c_5 \Big\} = 0$$