Aufgabe 1: Gitterwirkung

$$U_{\mu\nu}(x) = U_\nu^{-1}(x)\, U_\mu^{-1}(x+a\hat{\nu})\, U_\nu(x+a\hat{\mu})\, U_\mu(x)$$

$$U_\mu(x) = e^{-aG_\mu(x)} \qquad U_\mu^{-1}(x) = e^{aG_\mu(x)}$$

$$U_{\mu\nu}(x) = e^{aG_\nu(x)}\, e^{aG_\mu(x+a\hat{\nu})}\, e^{-aG_\nu(x+a\hat{\mu})}\, e^{-aG_\mu(x)}$$

$$= \exp\left\{a\left(G_\nu(x)+G_\mu(x+a\hat{\nu})\right) + \frac{a^2}{2}\left[G_\nu(x), G_\mu(x+a\hat{\nu})\right] + O(a^3)\right\}$$

$$* \exp\left\{-a\left(G_\nu(x+a\hat{\mu})+G_\mu(x)\right) + \frac{a^2}{2}\left[G_\nu(x+a\hat{\mu}), G_\mu(x)\right] + O(a^3)\right\}$$

$$= \exp\Big\{a\left[G_\nu(x)+G_\mu(x+a\hat{\nu}) - G_\nu(x+a\hat{\mu}) - G_\mu(x)\right]$$
$$+ a^2\left[G_\nu(x), G_\mu(x)\right] - \frac{a^2}{2}\left[G_\nu(x)+G_\mu(x), G_\nu(x)+G_\mu(x)\right] + O(a^3)\Big\}$$

$$= \exp\left\{a^2\left[\partial_\nu G_\mu(x) - \partial_\mu G_\nu(x)\right] + a^2\left[G_\nu(x), G_\mu(x)\right] + O(a^3)\right\}$$

$$U_{\mu\nu}(x) = \exp\left\{a^2 G_{\mu\nu}(x)\right\} \quad \text{mit } G_{\mu\nu} = \partial_\nu G_\mu - \partial_\mu G_\nu - [G_\mu, G_\nu]$$

$$U_{\mu\nu}^{-1}(x) = U_{\nu\mu}(x) = \exp\left\{-a^2 G_{\mu\nu}(x)\right\}$$

$$S = \frac{6}{g_s^2} \sum_x \sum_{1\le\mu<\nu\le4} \left\{1 - \frac{1}{6}\mathrm{Tr}\left[U_{\mu\nu}(x) + U_{\mu\nu}^{-1}(x)\right]\right\}$$

$$= \frac{6}{g_s^2} \sum_x \sum_{1\le\mu<\nu\le4} \left\{1 - \frac{1}{6}\mathrm{Tr}\left[1 + \cancelto{0}{a^2 G_{\mu\nu}} + \frac{a^4}{2}G_{\mu\nu}^2 + 1 + \cancelto{0}{a^2 G_{\nu\mu}} + \frac{a^4}{2}G_{\nu\mu}^2\right]\right\}$$

$$= \frac{6}{g_s^2} \sum_x \sum_{1\le\mu,\nu\le4} \left\{1 - 1 - \frac{a^4}{6}\mathrm{Tr}\, G_{\mu\nu}^2 + O(a^5)\right\} \qquad \sum_{1\le\mu<\nu\le4} = \frac{1}{2}\sum_{\mu,\nu}$$

$$= -\frac{1}{2g_s^2} \sum_x \sum_{\mu,\nu} a^4\, \mathrm{Tr}\, G_{\mu\nu}^2 \rightarrow -\frac{1}{2g_s^2}\int d^4x\, \mathrm{Tr}\, G_{\mu\nu}^2$$

Aufgabe 2: Richardson-Potential

$$V(r) = -\frac{4}{3}\frac{48\pi^2}{33-2N_F}\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{i\vec{q}\vec{r}}-1}{\vec{q}^2\log\left(1+\frac{\vec{q}^2}{\Lambda^2}\right)} + V(0)$$

$$V(r)-V(0) = -\frac{64\pi^2}{33-2N_F}\int_0^\infty\frac{dq}{(2\pi)^2}\int_{-1}^{1}dz\,\frac{e^{iqrz}-1}{\log\left(1+\frac{q^2}{\Lambda^2}\right)}$$

$$V(r)-V(0) = -\frac{16}{33-2N_F}\frac{2}{r}\int_0^\infty dq\,\frac{\sin qr-qr}{q\log\left(1+\frac{q^2}{\Lambda^2}\right)}$$

$$V(r)-V(0) = -\frac{32}{33-2N_F}\frac{1}{r}\int_0^\infty dx\,\frac{\sin x-x}{x\log\left(1+\frac{x^2}{t^2}\right)}$$

$$\boxed{\begin{array}{l} x=qr \\ t=\Lambda r\end{array}}$$

(i) $t \ll 1$:

$$V(r) = -\frac{32}{33-2N_F}\frac{1}{r}\int_0^\infty dx\,\frac{\sin x}{x\log\frac{x^2}{t^2}} + O(r^0)$$

$$= -\frac{16}{33-2N_F}\frac{1}{r}\int_0^\infty dx\,\frac{\sin x}{x}\frac{1}{\log x-\log t}$$

$$= \frac{16}{33-2N_F}\frac{1}{r\log t}\underbrace{\int_0^\infty dx\,\frac{\sin x}{x}}_{\frac{\pi}{2}} + O\left(\frac{1}{r\log^2 t}\right)$$

$$\boxed{V(r) \longrightarrow \frac{-8\pi}{33-2N_F}\frac{1}{r\log\left(\frac{1}{\Lambda r}\right)} = -\frac{4}{3}\frac{\alpha_s\left(\frac{1}{r^2}\right)}{r}}$$

$$\left(\alpha_s(q^2) = \frac{12\pi}{(33-2N_F)\log\frac{q^2}{\Lambda^2}}\right)$$

(iii) $t \gg 1$:

$$V(r) - V(0) \longrightarrow \frac{-32}{33-2N_F} \frac{t^2}{r} \int_0^\infty dx \, \frac{\sin x - x}{x^3}$$

$$= -\frac{32\Lambda^2 r}{33-2N_F} \left[\underbrace{\frac{1}{x} - \frac{\sin x}{2x^2} - \frac{\cos x}{2x} \Big|_0^\infty}_{=0} - \frac{1}{2} \underbrace{\int_0^\infty dx \, \frac{\sin x}{x}}_{=\frac{\pi}{2}} \right]$$

$$\boxed{V(r) \longrightarrow V(0) + \frac{8\pi\Lambda^2}{33-2N_F} r}$$