8. Übung

Aufgabe 1.

$$\int_0^1 dx\, x^{N-1} P_{qq}(x) = \frac{4}{3}\int_0^1 dx\,(x^{N-1}-1)\frac{1+x^2}{1-x}$$

$$= \frac{4}{3}\int_0^1 dx \left\{1+x-x^{N-1}-x^N-2\frac{1-x^{N-1}}{1-x}\right\}$$

$$= \frac{4}{3}\int_0^1 dx \left\{1+x-x^{N-1}-x^N-2\sum_{j=0}^{N-2}x^j\right\}$$

$$= \frac{4}{3}\left\{\frac{3}{2}-\frac{1}{N}-\frac{1}{N+1}-2\sum_{j=1}^{N-1}\frac{1}{j}\right\}$$

$$= \frac{4}{3}\left\{-\frac{1}{2}+\frac{1}{N(N+1)}-2\sum_{j=2}^{N}\frac{1}{j}\right\}$$

$$\int_0^1 dx\, x^{N-1} P_{qg}(x) = \frac{1}{2}\int_0^1 dx\left\{x^{N-1}-2x^N+2x^{N+1}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{N}-\frac{2}{N+1}+\frac{2}{N+2}\right\} = \frac{1}{2}\,\frac{N^2+N+2}{N(N+1)(N+2)}$$

$$\int_0^1 dx\, x^{N-1} P_{gq}(x) = \frac{4}{3}\int_0^1 dx\left\{2x^{N-2}-2x^{N-1}+x^N\right\} = \frac{4}{3}\left\{\frac{2}{N-1}-\frac{2}{N}+\frac{1}{N+1}\right\}$$

$$= \frac{4}{3}\,\frac{N^2+N+2}{(N-1)N(N+1)}$$

$$\int_0^1 dx\, x^{N-1} P_{gg}(x) = \frac{33-2N_F}{6}+6\int_0^1 dx\left\{-\frac{1-x^{N-1}}{1-x}+x^{N-2}-2x^{N-1}+x^N-x^{N+1}\right\}$$

$$= \frac{33-2N_F}{6}+6\left\{\frac{1}{N-1}-\frac{2}{N}+\frac{1}{N+1}-\frac{1}{N+2}-\sum_{j=0}^{N-2}\frac{1}{j+1}\right\}$$

$$= \frac{3}{2}\left\{\frac{4}{N(N-1)}+\frac{4}{(N+1)(N+2)}-\frac{1}{3}-4\sum_{j=2}^{N}\frac{1}{j}-\frac{2}{9}N_F\right\}$$

(2) Mellin-Inversion

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dN\, x^{-N} F(N) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dN\, x^{-N}\int_0^1 dy\, y^{N-1} f(y)$$

$$= \int_0^1 \frac{dy}{y} f(y) \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dN \left(\frac{y}{x}\right)^N \qquad N=\sigma+i\omega$$

$$= \int_0^1 \frac{dy}{y} f(y)\,\frac{1}{2\pi}\left(\frac{y}{x}\right)^\sigma \underbrace{\int_{-\infty}^{\infty} d\omega\, e^{i\omega \log\frac{y}{x}}}_{=2\pi\,\delta_1\left(\log\frac{y}{x}\right) = 2\pi x\,\delta_1(y-x)}$$

$$= \int_0^1 \frac{dy}{y} f(y)\,\frac{1}{2\pi}\left(\frac{y}{x}\right)^\sigma 2\pi x\,\delta_1(x-y) = f(x)$$

$$\int_0^1 dx\, x^{N-1}(f\otimes g)(x) = \int_0^1 dx \int_x^1 \frac{dz}{z}\, x^{N-1} f(z)\, g\left(\frac{x}{z}\right)$$

$$= \int_0^1 \frac{dz}{z}\int_0^z dx\, x^{N-1} f(z)\, g\left(\frac{x}{z}\right) = \int_0^1\frac{dz}{z}\int_0^1 dy\, z\,(zy)^{N-1} f(z)\, g(y)$$

$$= \int_0^1 dz\, z^{N-1} f(z)\int_0^1 dy\, y^{N-1} g(y) = F(N)\,G(N)$$