Übung 4 (Musterlösung)

$$\frac{\partial \alpha_s}{\partial t} = \beta(\alpha_s) = b_0 \alpha_s^2 + b_1 \alpha_s^3 + b_2 \alpha_s^4 + \dots \qquad t = \log\frac{\mu}{\mu_0}$$

$$\Longleftrightarrow \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} = t$$

$$t = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{b_0\alpha_s^2}\left\{1 - \frac{b_1\alpha_s^3 + b_2\alpha_s^4}{b_0\alpha_s^2} + \frac{b_1^2\alpha_s^6}{b_0^2\alpha_s^4} + O(\alpha_s)\right\}$$

$$b_0 t = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \left\{\frac{1}{\alpha_s^2} - \frac{b_1 + b_2\alpha_s}{b_0\alpha_s} + \frac{b_1^2}{b_0^2} + O(\alpha_s)\right\}$$

$$= -\frac{1}{\alpha_s(\mu)} + \frac{1}{\alpha_s(\mu_0)} - \frac{b_1}{b_0}\log\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} - \frac{b_2 b_0 - b_1^2}{b_0^2}\left[\alpha_s(\mu) - \alpha_s(\mu_0)\right] + O(\alpha_s^2)$$

$$\Longleftrightarrow \boxed{\frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_s(\mu_0)} - b_0\log\frac{\mu}{\mu_0} - \frac{b_1}{b_0}\log\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} - \frac{b_2 b_0 - b_1^2}{b_0^2}\left[\alpha_s(\mu) - \alpha_s(\mu_0)\right] + O(\alpha_s^2)}$$

Stammfunktion:

$$b_0 \log\frac{\mu}{\Lambda} = -\frac{1}{\alpha_s(\mu)} - \frac{b_1}{b_0}\log\left(\frac{b_0\alpha_s(\mu)}{-2}\right) - \frac{b_2 b_0 - b_1^2}{b_0^2}\alpha_s(\mu) - C + O(\alpha_s^2)$$

$$\Longrightarrow \boxed{\Lambda = \mu \exp\left\{\frac{1}{b_0\alpha_s(\mu)} - \frac{b_1}{b_0^2}\log\left(\frac{-2}{b_0\alpha_s(\mu)}\right) + \frac{b_2 b_0 - b_1^2}{b_0^3}\alpha_s(\mu) + C + O(\alpha_s^2)\right\}}$$

$$\Longleftrightarrow \frac{1}{\alpha_s(\mu)} = -\frac{b_0}{2}L + \frac{b_1}{b_0}\log\left(\frac{-2}{b_0\alpha_s(\mu)}\right) - b_0 C - \frac{b_2 b_0 - b_1^2}{b_0^2}\alpha_s(\mu) + O(\alpha_s^2)$$

$$L = \log\frac{\mu^2}{\Lambda^2}$$

Iteration: $\displaystyle \frac{1}{\alpha_s(\mu)} = -\frac{b_0}{2}L + \Delta_1 = -\frac{b_0}{2}L\left(1 - \frac{2}{b_0}\frac{\Delta_1}{L}\right)$

$$\Delta_1 = \frac{b_1}{b_0}\left[\log L - \frac{2}{b_0}\frac{\Delta_1}{L}\right] - b_0 C - \frac{b_2 b_0 - b_1^2}{b_0^2}\left[-\frac{2}{b_0 L}\right] + O\left(\frac{1}{L^2}\right)$$

$$\Longleftrightarrow \Delta_1 = \left\{ \frac{b_1}{b_0} \log L - b_0 C + 2\,\frac{b_2 b_0 - b_1^2}{b_0^3 L} \right\} \left\{ 1 - \frac{2 b_1}{b_0 L} \right\} + O\!\left(\frac{1}{L^2}\right)$$

$$= \frac{b_1}{b_0}\left[\log L - \frac{b_0^2}{b_1} C\right] - 2\,\frac{b_1^2}{b_0^3}\,\frac{\log L - \frac{b_0^2}{b_1} C - \left(\frac{b_2 b_0}{b_1^2} - 1\right)}{L} + O\!\left(\frac{1}{L^2}\right)$$

$$\Longrightarrow \boxed{\frac{1}{\alpha_s(\mu)} = -\frac{b_0}{2}\log\frac{\mu^2}{\Lambda^2} + \frac{b_1}{b_0}\left[\log\log\frac{\mu^2}{\Lambda^2} - \frac{b_0^2}{b_1} C\right] - 2\,\frac{b_1^2}{b_0^3}\,\frac{\log\log\frac{\mu^2}{\Lambda^2} - \frac{b_0^2}{b_1} C - \left(\frac{b_2 b_0}{b_1^2} - 1\right)}{\log\frac{\mu^2}{\Lambda^2}} + O\!\left(\frac{1}{\log^2\frac{\mu^2}{\Lambda^2}}\right)}$$

$$\alpha_s(\mu^2) = -\frac{2}{b_0 L}\left\{ 1 + \frac{b_1}{b_0}\left[\log L - \frac{b_0^2}{b_1} C\right]\frac{2}{b_0 L} - \frac{b_1^2}{b_0^2}\left[\log L - \frac{b_0^2}{b_1} C\right]^2 \frac{4}{b_0^4 L^2} - 4\,\frac{b_1^2}{b_0^4}\,\frac{\log L - \frac{b_0^2}{b_1} C - \frac{b_2 b_0}{b_1^2} + 1}{L^2} + O\!\left(\frac{1}{L^3}\right) \right\}$$

$$\boxed{\alpha_s(\mu^2) = -\frac{2}{b_0 \log\frac{\mu^2}{\Lambda^2}}\left\{ 1 + 2\,\frac{b_1}{b_0^2}\,\frac{\log\log\frac{\mu^2}{\Lambda^2} - \frac{b_0^2}{b_1} C}{\log\frac{\mu^2}{\Lambda^2}} + 4\,\frac{b_1^2}{b_0^2 \log^2\frac{\mu^2}{\Lambda^2}}\left[\left(\log\log\frac{\mu^2}{\Lambda^2} - \frac{b_0^2}{b_1} C - \frac{1}{2}\right)^2 + \frac{b_2 b_0}{b_1^2} - \frac{5}{4}\right] + O\!\left(\frac{1}{\log^3\frac{\mu^2}{\Lambda^2}}\right) \right\}}$$

$$b_0 = -\frac{1}{2\pi}\left(11 - \frac{2}{3} N_F\right) = -\frac{1}{\pi}\left(\frac{11}{2} - \frac{N_F}{3}\right)$$

$$b_1 = -\frac{1}{\pi^2}\left(\frac{51}{4} - \frac{19}{12} N_F\right)$$

$$b_2 = -\frac{1}{\pi^3}\left(\frac{2857}{64} - \frac{5033}{576} N_F + \frac{325}{1728} N_F^2\right)$$

N_F	πb_0	$\pi^2 b_1$	$\pi^3 b_2$
3	$-\frac{9}{2}$	-8	$\frac{3863}{192}$
4	$-\frac{25}{6}$	$-\frac{77}{12}$	$-\frac{21943}{1728}$
5	$-\frac{23}{6}$	$-\frac{29}{6}$	$-\frac{9769}{1728}$
6	$-\frac{7}{2}$	$-\frac{13}{4}$	$+\frac{65}{64}$

$$\boxed{\begin{aligned}
\Lambda^{(6)}_{\overline{MS}} &= \mu \exp\left\{-\frac{2}{7}\left[\frac{\pi}{\alpha_s(\mu)} - \frac{13}{14}\log\left(\frac{4\pi}{7\alpha_s(\mu)}\right)\right]\right\} \\
\Lambda^{(5)}_{\overline{MS}} &= \mu \exp\left\{-\frac{6}{23}\left[\frac{\pi}{\alpha_s(\mu)} - \frac{29}{23}\log\left(\frac{12\pi}{23\alpha_s(\mu)}\right)\right]\right\} \\
\Lambda^{(4)}_{\overline{MS}} &= \mu \exp\left\{-\frac{6}{25}\left[\frac{\pi}{\alpha_s(\mu)} - \frac{77}{50}\log\left(\frac{12\pi}{25\alpha_s(\mu)}\right)\right]\right\} \\
\Lambda^{(3)}_{\overline{MS}} &= \mu \exp\left\{-\frac{2}{9}\left[\frac{\pi}{\alpha_s(\mu)} - \frac{16}{9}\log\left(\frac{4\pi}{9\alpha_s(\mu)}\right)\right]\right\}
\end{aligned}}$$

$$\alpha_s^{-1}(m_q) = -\frac{b_0(N_F+1)}{2}\log\frac{m_q^2}{\Lambda^{(N_F+1)}} + \frac{b_1(N_F+1)}{b_0(N_F+1)}\log\log\frac{m_q^2}{\Lambda^{(N_F+1)}}$$

$$= -\frac{b_0(N_F)}{2}\log\frac{m_q^2}{\Lambda^{(N_F)}} + \frac{b_1(N_F)}{b_0(N_F)}\log\log\frac{\mu^2}{\Lambda^{(N_F)}}$$

generically $\quad -\frac{b_0'}{2}L' + \frac{b_1'}{b_0'}\log L' = -\frac{b_0}{2}L + \frac{b_1}{b_0}\log L$

$$L' = \frac{b_0}{b_0'}L + X \qquad\qquad L = \log\frac{m_q^2}{\Lambda^2}$$

$$\Rightarrow -\frac{b_0'}{2}X + \frac{b_1'}{b_0'}\left[\log\frac{b_0}{b_0'}L + \frac{b_0'}{b_0}\frac{X}{L}\right] = \frac{b_1}{b_0}\log L$$

$$X\left[-\frac{b_0'}{2} + \frac{b_1'}{b_0}\frac{1}{L}\right] = \left(\frac{b_1}{b_0} - \frac{b_1'}{b_0'}\right)\log L - \frac{b_1'}{b_0'}\log\frac{b_0}{b_0'}$$

$$X = -\frac{2}{b_0'}\left(\frac{b_1}{b_0} - \frac{b_1'}{b_0'}\right)\log L$$

$$L' = \frac{b_0}{b_0'}L - \frac{2}{b_0'}\left(\frac{b_1}{b_0} - \frac{b_1'}{b_0'}\right)\log L$$

$$\frac{m_q^2}{\Lambda'^2} = \left(\frac{m_q^2}{\Lambda^2}\right)^{\frac{b_0}{b_0'}} \left[\log\left(\frac{m_q^2}{\Lambda^2}\right)\right]^{\frac{2}{b_0'}\left(\frac{b_1}{b_0}-\frac{b_1'}{b_0'}\right)}$$

$$\Lambda' = \Lambda \left(\frac{\Lambda}{m_q}\right)^{\frac{b_0}{b_0'}-1} \left[\log\left(\frac{m_q^2}{\Lambda^2}\right)\right]^{-\frac{1}{b_0'}\left(\frac{b_1'}{b_0'}-\frac{b_1}{b_0}\right)}$$

$$\Lambda_6 = \Lambda_5 \left(\frac{\Lambda_5}{m_t}\right)^{\frac{2}{21}} \left[\log \frac{m_t^2}{\Lambda_5^2}\right]^{-\frac{321}{3381}}$$

$$\Lambda_5 = \Lambda_6 \left(\frac{m_t}{\Lambda_6}\right)^{\frac{2}{23}} \left[\log \frac{m_t^2}{\Lambda_6^2}\right]^{\frac{321}{3703}}$$

$$\Lambda_5 = \Lambda_4 \left(\frac{\Lambda_4}{m_b}\right)^{\frac{2}{23}} \left[\log \frac{m_b^2}{\Lambda_4^2}\right]^{-\frac{963}{13225}}$$

$$\Lambda_4 = \Lambda_5 \left(\frac{m_b}{\Lambda_5}\right)^{\frac{2}{25}} \left[\log \frac{m_b^2}{\Lambda_5^2}\right]^{\frac{963}{14375}}$$

$$\Lambda_4 = \Lambda_3 \left(\frac{\Lambda_3}{m_c}\right)^{\frac{2}{25}} \left[\log \frac{m_c^2}{\Lambda_3^2}\right]^{-\frac{107}{1875}}$$

$$\Lambda_3 = \Lambda_4 \left(\frac{\Lambda_4}{m_c}\right)^{\frac{2}{27}} \left[\log \frac{m_c^2}{\Lambda_4^2}\right]^{\frac{107}{2025}}.$$

$\Lambda = \Lambda_4$ als Standard [mit Konstante $C_4 = 0$] $L = \log \frac{\mu^2}{\Lambda^2}$

$m_s < \mu < m_c$: $\alpha_s(\mu) = \frac{12\pi}{27 L}\left\{1 - \frac{\frac{64}{81}\log L + 2C_3}{L}\right\}$

$m_c < \mu < m_b$: $\alpha_s(\mu) = \frac{12\pi}{25 L}\left\{1 - \frac{\frac{462}{625}\log L + 2C_4}{L}\right\}$

$m_b < \mu < m_t$: $\alpha_s(\mu) = \frac{12\pi}{23 L}\left\{1 - \frac{\frac{348}{529}\log L + 2C_5}{L}\right\}$

$$C_3 = \log\frac{\Lambda_4}{\Lambda_3} = -\frac{1}{27}\left\{\log\frac{m_c^2}{\Lambda_4^2} + \frac{107}{75}\log\log\frac{m_c^2}{\Lambda_4^2}\right\}$$

$$C_4 = 0$$

$$C_5 = \log\frac{\Lambda_4}{\Lambda_5} = \frac{1}{23}\left\{\log\frac{m_b^2}{\Lambda_4^2} + \frac{963}{575}\log\log\frac{m_b^2}{\Lambda_4^2}\right\}$$