Musterlösung

1. Aufgabe: $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$ falls $[A,B] \in \mathbb{C}$

$$\Longrightarrow e^B e^A = e^A e^B e^{-[A,B]}$$

$\varphi(x) = \varphi_+(x) + \varphi_-(x)$ $\qquad \varphi_+(x)|0\rangle = 0$

Vernichter Erzeuger $\qquad \langle 0|\varphi_-(x) = 0$

Diskretisierung der Zeit: $f^k \equiv \int_{t=t_k} d^3x\, f(x)$

$$T e^{i\int d^4x\, j(x)\varphi(x)} \Longleftarrow T e^{i\sum_k j^k[\varphi_+^k + \varphi_-^k]}$$

$$T e^{i\sum_k j^k(\varphi_+^k+\varphi_-^k)} = \prod_{\substack{k \\ t_{k+1}>t_k}} e^{ij^k(\varphi_+^k+\varphi_-^k)} = \prod_{\substack{k \\ t_{k+1}>t_k}} e^{ij^k\varphi_-^k} e^{ij^k\varphi_+^k} \quad (*)$$

denn: $[\varphi_+(x), \varphi_-(y)]_{x^0=y^0} = 0 \Longrightarrow [\varphi_+^k, \varphi_-^k] = 0$

$$(*) = \prod_k e^{ij^k\varphi_-^k} \prod_l e^{ij^l\varphi_+^l} \prod_{\substack{m,k \\ t_m>t_k}} e^{-i^2[\varphi_-^k, \varphi_+^m]j^m j^k}$$

$$\Rightarrow :e^{i\int d^4x\, j(x)\varphi(x)}: e^{-\int d^4x\, d^4y\, j(x)j(y)\Theta(x^0-y^0)[\varphi_+(x), \varphi_-(y)]}$$

$$i\Delta_F(x) = \langle 0|T\{\varphi(x)\varphi(0)\}|0\rangle = \langle 0|\{\Theta(x^0)\varphi_+(x)\varphi_-(0) + \Theta(-x^0)\varphi_+(0)\varphi_-(x)\}|0\rangle$$

$$= \Theta(x^0)\langle 0|\varphi_+(x)\varphi_-(0)|0\rangle + \Theta(-x^0)\langle 0|\varphi_+(-x)\varphi_-(0)|0\rangle$$

$$\Rightarrow \int d^4x\, \langle 0|T\{\varphi(x), \varphi(0)\}|0\rangle = 2\int d^4x\, \Theta(x^0)\langle 0|[\varphi_+(x), \varphi_-(0)]|0\rangle = 2\int d^4x\, \Theta(x^0)[\varphi_+(x), \varphi_-(0)]$$

$$\Longrightarrow \boxed{W_0(j) = \langle 0|T e^{i\int d^4x\, j(x)\varphi(x)}|0\rangle = e^{-\frac{1}{2}\int d^4x\, d^4y\, j(x)\Delta_F(x-y)j(y)}}$$

2. Aufgabe:

$$\eta = M \xi$$

$$f(\eta) = \dots + f_1 \eta_1 \cdots \eta_n$$

$$\eta_1 \cdots \eta_n = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} M_{ij} \xi_j \right) = M_{1j_1} \cdots M_{nj_n} \underbrace{\xi_{j_1} \cdots \xi_{j_n}}_{= \varepsilon_{j_1 \dots j_n} \xi_1 \cdots \xi_n}$$

$$= \varepsilon_{j_1 \dots j_n} M_{1j_1} \cdots M_{nj_n} \xi_1 \cdots \xi_n = (\mathrm{Det} M)\, \xi_1 \cdots \xi_n$$

$$\Longrightarrow \boxed{d\eta_1 \cdots d\eta_n = \mathrm{Det} M^{-1}\, d\xi_1 \cdots d\xi_n}$$

$$\xi = Q\eta \Longrightarrow \eta = Q^{-1}\xi \Longrightarrow \mathcal{D}\eta = \mathrm{Det} Q\, \mathcal{D}\xi$$

$$\int \mathcal{D}\bar\eta\, \mathcal{D}\eta\, e^{-\bar\eta Q \eta} = \mathrm{Det} Q \int \mathcal{D}\bar\eta\, \mathcal{D}\xi\, e^{-\bar\eta \xi}$$

$$\Longrightarrow \boxed{\frac{\int \mathcal{D}\bar\eta\, \mathcal{D}\eta\, e^{-\bar\eta Q \eta}}{\int \mathcal{D}\bar\eta\, \mathcal{D}\eta\, e^{-\bar\eta \eta}} = \mathrm{Det} Q}$$

$$e^{-\bar\eta \xi} = \sum_{i=1}^{\infty} \frac{(-\bar\eta_i \xi_i)^i}{i!} = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \sum_{k_1 + \dots + k_n = i} (\bar\eta_1 \xi_1)^{k_1} \cdots (\bar\eta_n \xi_n)^{k_n} \frac{i!}{k_1! \cdots k_n!}$$

$$= (-1)^n (\bar\eta_1 \xi_1) \cdots (\bar\eta_n \xi_n)$$

$$\eta_i \bar\eta_j = -Q^{-1}_{ik} \bar\eta_j \xi_k$$

$$\Longrightarrow \eta_i \bar\eta_j e^{-\bar\eta Q \eta} = -Q^{-1}_{ik} \delta_{kj} \bar\eta_j \xi_j (-1)^{n-1} (\bar\eta_1 \xi_1) \cdots (\bar\eta_{j-1} \xi_{j-1}) (\bar\eta_{j+1} \xi_j) \cdots (\bar\eta_n \xi_n)$$

$$= Q^{-1}_{ij} (-1)^n (\bar\eta_1 \xi_1) \cdots (\bar\eta_n \xi_n) = Q^{-1}_{ik} e^{-\bar\eta \xi}$$

$$\Longrightarrow \boxed{\frac{\int \mathcal{D}\bar\eta\, \mathcal{D}\eta\, \eta_i \bar\eta_j e^{-\bar\eta Q \eta}}{\int \mathcal{D}\bar\eta\, \mathcal{D}\eta\, e^{-\bar\eta \eta}} = Q^{-1}_{ij}\, \mathrm{Det} Q}$$