QUANTUM CHROMODYNAMICS

Program

A. Basics of QCD:
   color d.o.f. of quarks
   non-Abelian field theory of quarks and gluons
   asymptotic freedom

B. QCD @ short distances:
   nucleon structure functions
   e^+e^- annihilation into hadrons
   Drell–Yan processes
   jet physics in e^+e^- annihilation and hadron-hadron scattering
   quarkonium physics
   soft gluon resummation

C. QCD @ large distances:
   lattice gauge theory of QCD
   QCD vacuum

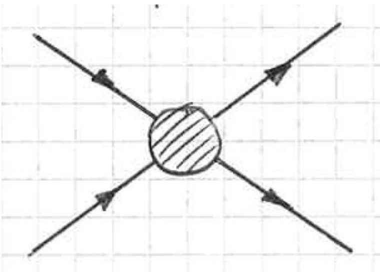
web page:
`http://tiger.web.psi.ch/vorlesung/qcd/`

A. BASICS OF QCD

§1. Introduction of Color

QCD: field theoretical formulation of strong int.
historical definition of strong interactions:

- binding force of nucleons inside nucleus
- force in nucleon-nucleon scattering



interaction distance: $d \sim 1$ fm

$\rightarrow \sigma \sim 4\pi d^2 \sim 10$ mb

interaction strength: $V(R) = \frac{g_s^2}{4\pi}\, e^{-\frac{R}{d}}$

$\frac{g_s^2}{4\pi} \sim 100 \frac{g_{elm}^2}{4\pi} \sim 1$

Spin-statistics problem of the quark model

$\Delta^{++}\left(s_z = \frac{3}{2}\right) = u(\uparrow)u(\uparrow)u(\uparrow) \leftarrow$ totally symm. spin wave funct.
$\in$ decuplet

$\uparrow$ Fermi statistics: totally antisymm. wave function

(i) ground state $\neq$ rel. S-wave combination
P-waves $\rightarrow$ knots $\rightarrow$ forbidden zones
$\rightarrow$ larger energy due to uncertainty principle
↯ to naive experience

(ii) magnetic moments of nucleons

$\vec{\mu} = \frac{eQ}{2m}\left[\vec{\ell} + 2\vec{S}\right]$

S-waves $\ell = 0$: nucleon moments are built up additively from quark moments

$$\mu_{\mathcal{N}} = \langle \mathcal{N} | \sum_{i=1}^{3} \mu(i)\sigma_3(i) | \mathcal{N} \rangle$$

due to the spin wave function: $[\mu_u = -2\mu_d]$

$$\mu_p = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d = -\left(\frac{8}{3} + \frac{1}{3}\right)\mu_d = -3\mu_d \text{ for } m_u \approx m_d$$

↑ Clebsch–Gordan

$$\mu_n = \frac{4}{3}\mu_d - \frac{1}{3}\mu_u = \left(\frac{4}{3} + \frac{2}{3}\right)\mu_d = 2\mu_d$$

ratio: $\boxed{\frac{\mu_p}{\mu_n} = -\frac{3}{2} \qquad \text{exp} = -1.46}$

no $\ell \neq 0$ contribution required

effective quark mass:

$$\mu_p = \frac{e}{2m_p} 2.79 = -\frac{1}{3}\frac{e}{2m_d}(-3) = \frac{e}{2m_d}$$

$$\Rightarrow \boxed{m_q^{eff} = \frac{m_p}{2.79} \approx 330 \text{ MeV}}$$

Solution: quarks carry 3-valued differentiator so that symmetric quark model possible

I. Color Hypothesis (Greenberg ’64)

Next to flavor charges quarks carry color charges; each quark appears in exactly 3 colors (red, blue, green = 1,2,3): $q = (q_1, q_2, q_3)$

color transformations: maximal mixing group of the 3 color d.o.f. ($\neq$ common phase)

$q \to q' = e^{-i \sum_{k=1}^{8} \alpha_k \frac{\lambda_k}{2}}\, q \leftarrow SU(3)_{\mathcal{C}}$ transformations
= unimodular, unitary 3×3 matrices
[non-Abelian group]

<u>Gell-Mann matrices:</u> $\lambda_k \quad k = 1, 2, \ldots, 8$
(3-dim. extension of $\vec{\sigma}$ in $SU(2)$)

$\lambda_k^\dagger = \lambda_k \quad \Rightarrow \quad e^{-i\alpha_k \frac{\lambda_k}{2}}$ unitary: $U^\dagger U = \mathbb{1}$
$Tr \lambda_k = 0 \quad \Rightarrow \quad$ unimodular: Det $U = +1$
<u>explicit representation:</u>

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

properties: $T^k = \frac{\lambda_k}{2}$

$$
\begin{aligned}
[T^a, T^b] &= i f_{abc} T^c \qquad [A_2 \text{ algebra}] \\
\{T^a, T^b\} &= \frac{1}{3}\delta_{ab}\mathbb{1} + d_{abc}T^c \\
Tr(T^a T^b) &= \frac{1}{2}\delta_{ab} \qquad Tr(T^a) = 0
\end{aligned}
$$

I'. Color Hypothesis (Gell-Mann '72)

The $SU(3)_{\mathcal{C}}$ symmetry is exact. All physical (free) states, observables and int. are $SU(3)_{\mathcal{C}}$ singlets.

(a) quarks as color triplets do not appear as free particles.

(b) color wave functions:

$$
\left.\begin{array}{l}
\text{baryon: } \frac{1}{\sqrt{6}}\epsilon_{ijk} \\
\text{meson: } \frac{1}{\sqrt{3}}\delta_{ij}
\end{array}\right\} \epsilon_{ijk}, \delta_{ij} \;\; SU(3)_{\mathcal{C}} \text{ singlets}
$$

$$
\begin{aligned}
\text{Ex.: } \Delta^{++}\left(s_z = \frac{3}{2}\right) &= \frac{1}{\sqrt{6}}\epsilon_{ijk}u_i(\uparrow)u_j(\uparrow)u_k(\uparrow) \\
\Phi(s_z = +1) &= \frac{1}{\sqrt{3}}\delta_{ij}s_i(\uparrow)\bar{s}_j(\uparrow)
\end{aligned}
$$

(c) elm. int.: $\mathcal{L}_{elm} = -e j^\mu A_\mu$

$$
j_\mu = \sum_{fl} \bar{q}\gamma_\mu Q_q q \equiv \sum_{fl}\sum_{c} \bar{q}_c\gamma_\mu Q_q q_c
$$

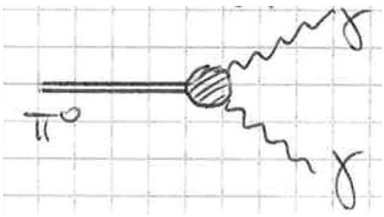
$SU(3)_{\mathcal{C}}$ singlet

The Nonvanishing Values of f_{ijk} and d_{ijk}

(ijk)	f_{ijk}	(ijk)	d_{ijk}
123	1	118	$1/\sqrt{3}$
147	$\frac{1}{2}$	146	$\frac{1}{2}$
156	$-\frac{1}{2}$	157	$\frac{1}{2}$
246	$\frac{1}{2}$	228	$1/\sqrt{3}$
257	$\frac{1}{2}$	247	$-\frac{1}{2}$
345	$\frac{1}{2}$	256	$\frac{1}{2}$
367	$-\frac{1}{2}$	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	$\frac{1}{2}$
678	$\sqrt{3}/2$	355	$\frac{1}{2}$
		366	$-\frac{1}{2}$
		377	$-\frac{1}{2}$
		448	$-1/2\sqrt{3}$
		558	$-1/2\sqrt{3}$
		668	$-1/2\sqrt{3}$
		778	$-1/2\sqrt{3}$
		888	$-1/\sqrt{3}$

TESTS OF THE COLOR HYPOTHESIS

1.) $\pi^0 \to \gamma\gamma$ decay

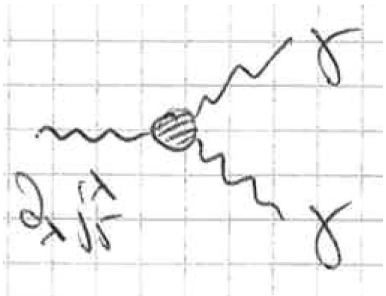


$$\mathcal{M}(\pi^0 \to \gamma\gamma) = i\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)T_{\mu\nu}(k_1, k_2; p)$$

Lorentz inv.
parity inv. } $T_{\mu\nu}(k_1, k_2; p) = \epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta T(p^2 = m_\pi^2)$

[π^0 = pseudoscalar]

Green's fct.:



$j_5^\lambda = \bar{q}\gamma^\lambda\gamma_5\frac{\lambda^3}{2}q$ $\qquad$ $\partial_\lambda j_5^\lambda \sim \pi^0$ quantum numbers

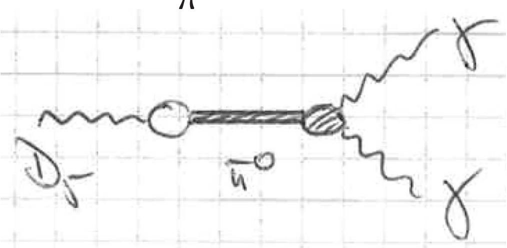
$= \frac{1}{2}\bar{u}\gamma^\lambda\gamma_5 u - \frac{1}{2}\bar{d}\gamma^\lambda\gamma_5 d$ $\qquad$ color-summed

$= \epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta A(p^2)$: $\qquad$ $p_\lambda\langle j_5^\lambda j^\mu j^\nu\rangle \sim \langle\partial_\lambda j_5^\lambda j^\mu j^\nu\rangle$

properties of $A(p^2)$:

(i) $A(p^2 = 0) = 0$ $\qquad$ [from $p_\lambda\times$decomposition $\langle j_5^\lambda j^\mu j^\nu\rangle$]

(ii) $p^2 \to m_\pi^2$:

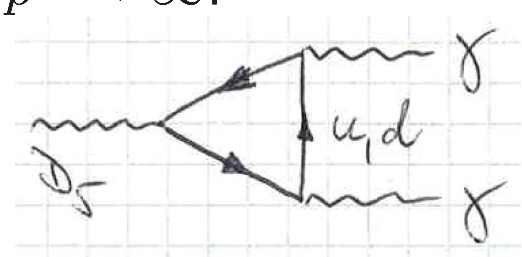


$$= -\frac{m_\pi^2\frac{f_\pi}{\sqrt{2}}T(\pi^0 \to \gamma\gamma)}{p^2 - m_\pi^2} \qquad \text{[PCAC]}$$

(iii) intermediate multi-particle states $= \sum\!\!\!\!\!\!\!\int_{(3m_\pi)^2}$

$\leftarrow$ negligible for $p^2 \to 0$ [$\sim$ 10%]

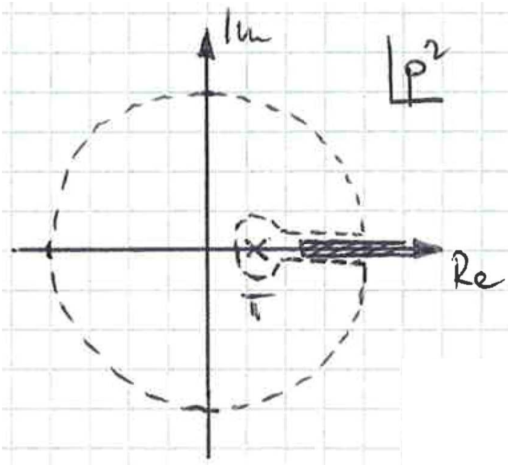
(iv) $p^2 \to \infty$:



$$= -\frac{e^2}{2\pi^2}\frac{1}{2}(Q_u^2 - Q_d^2)\underline{\underline{\underline{N_\mathcal{C}}}}$$

$N_\mathcal{C}$ = # colors

dispersion relation for $A(p^2)$:



Cauchy: $A(0) = 0 = \oint dp^2 \frac{A(p^2)}{p^2}$

$$= \int_{cut} \frac{dp^2}{p^2} \frac{m_\pi^2 \frac{-f_\pi}{\sqrt{2}} T}{p^2 - m_\pi^2} + 2\pi i A_\infty$$

$$0 = \frac{f_\pi}{\sqrt{2}} T(\pi^0 \to \gamma\gamma) + \frac{e^2}{4\pi^2}(Q_u^2 - Q_d^2) N_C$$

Width: $$\boxed{\Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha^2}{32\pi^3} \frac{m_\pi^3}{f_\pi^2} (Q_u^2 - Q_d^2)^2 N_C^2}$$

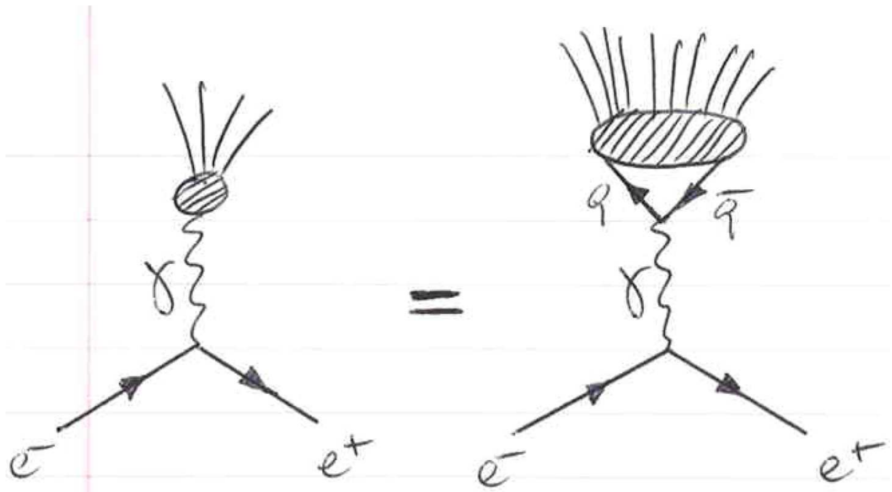
w/o color $N_C = 1$: $\Gamma = 0.868 \pm 0.065$ eV

w/ color $N_C = 3$: $\Gamma = 7.81 \pm 0.60$ eV ←

experimental: $\Gamma_{exp} = 7.84 \pm 0.56$ eV ←

2.) $e^+e^- \to$ hadrons

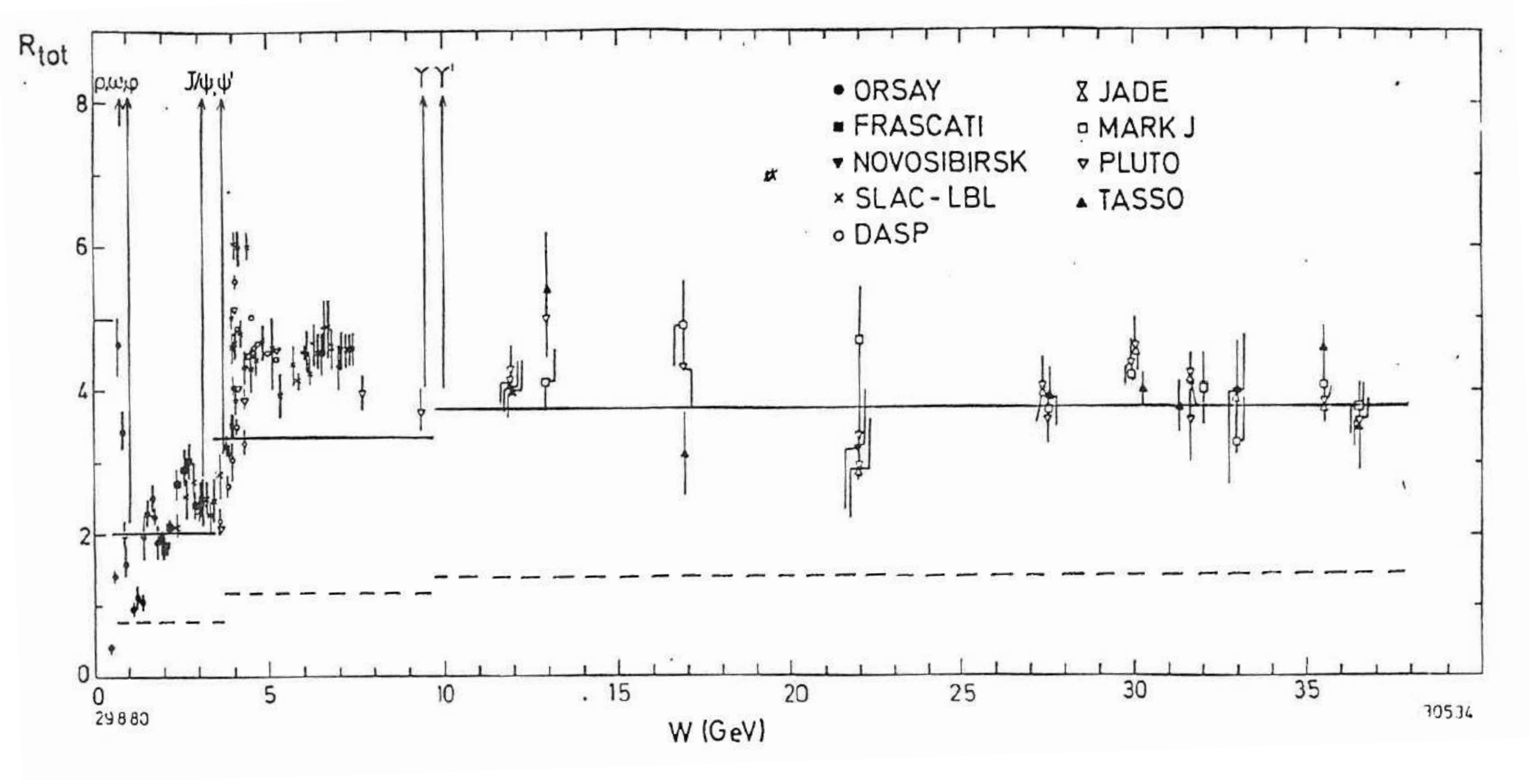
In the quark-parton model the production probability in $e^+e^- \to$ hadrons is determined by the one for $q\bar{q}$ pairs; final-state interactions are negligible for $\frac{d_{prod\ q\bar{q}}}{d_{hadron}} \sim \frac{1\ \text{GeV}}{E} \to 0 \quad (E \to \infty)$.



$$R = \frac{\sigma(e^+e^- \to \text{had.})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{fl,c} \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{fl} e_q^2$$

q	e_q
u, c, t	$+\frac{2}{3}$
d, s, b	$-\frac{1}{3}$

energy	prod. q's	R w/o color	R w/ color
< 3 GeV	u, d, s	$\frac{4}{9}+\frac{1}{9}+\frac{1}{9}=\frac{2}{3}$	2
> 5 GeV	$+c$	$\frac{6}{9}+\frac{4}{9}=\frac{10}{9}$	$\frac{10}{3}$
> 10 GeV	$+b$	$\frac{10}{9}+\frac{1}{9}=\frac{11}{9}$	$\frac{11}{3}$



§2. Gluon Gauge Fields

In analogy to QED:

II. Color Hypothesis (Nambu '66, Fritzsch+Gell-Mann '72 Leutwyler '73)
Color charges are sources of gauge fields ($\Rightarrow$ gluons) that build up the strong interaction between quarks.

Lagrangian for color triplet:

$$\mathcal{L}_q = \bar{q}(x)(i\partial\!\!\!/ - m_q)q(x) \quad \text{with } q = (q_1, q_2, q_3)$$
$$m_{q_1} = m_{q_2} = m_{q_3} \; SU(3)_{\mathcal{C}} \text{ singlet}$$

— invariant w.r.t. global, non-Abelian $SU(3)_{\mathcal{C}}$ transformations

$$\left.\begin{array}{l} q(x) \to Sq(x) \\ \bar{q}(x) \to \bar{q}(x)S^{-1} \end{array}\right\} S = e^{-i\alpha_k T^k} \qquad \left(T^k = \frac{\lambda_k}{2}\right)$$

— not invariant w.r.t. local $SU(3)_{\mathcal{C}}$ transformations: $\alpha_k = \alpha_k(x)$

$$\mathcal{L}_q \to \mathcal{L}_q + \bar{q}(x)\left(S^{-1} i\partial\!\!\!/ S\right) q(x)$$

made locally gauge invariant by introducing 8 minimally coupled gluon fields $G^k_\mu(x)$ ($k = 1, \ldots, 8$) (gluon matrix $G_\mu = G^k_\mu T^k$)

$$i\partial_\mu \to i\partial_\mu - g_s G_\mu = iD_\mu$$
$$\mathcal{L}_q = \bar{q}(x)(iD\!\!\!\!/ - m_q)q(x) = \bar{q}(x)[i\partial\!\!\!/ - m_q - g_s G\!\!\!\!/(x)]q(x)$$

with
$$q(x) \to S(x)q(x) \qquad \alpha_k = \alpha_k(x)$$
$$\bar{q}(x) \to \bar{q}(x)S^{-1}$$
$$G_\mu(x) \to SG_\mu S^{-1} - \frac{i}{g_s} S\partial_\mu S^{-1}$$

ROT. TRANSL.

covar. deriv.: $iDq \to iD'q' = \left[i\partial - g_s SGS^{-1} - i(\partial S)S^{-1}\right] Sq$

$\left[\partial(SS^{-1}) = 0\right]$ $\qquad = S(i\partial - g_s G)q = SiDS^{-1}Sq$

$\underline{\underline{D \to D' = SDS^{-1}}}$ ROTATION

<u>Gluon field Lagrangian:</u>

curl: $G_{\mu\nu} = D_\nu G_\mu - D_\mu G_\nu = \partial_\nu G_\mu - \partial_\mu G_\nu - ig_s[G_\mu, G_\nu]$

gauge trf.: $G_{\mu\nu} \to G'_{\mu\nu} = SG_{\mu\nu}S^{-1}$ pure rotation

from $G_{\mu\nu} = \frac{i}{g_s}[D_\mu, D_\nu]$ [no observable]

$\underline{\underline{\mathcal{L}_g = -\frac{1}{2} TrG^2_{\mu\nu} = -\frac{1}{4}(G^k_{\mu\nu})^2}}$ ← gauge invariant:

↑ no mass term $\left(+\frac{1}{2}m_g^2 TrG^2_\mu\right)$

consists of: (a) kinetic part $= -\frac{1}{4}(\partial_\nu G^k_\mu - \partial_\mu G^k_\nu)^2$

(b) trilinear coupling $\sim g_s GGG$

(c) quartic coupling $\sim g_s^2 GGGG$

— self-interaction of gluon fields: color-charged gluons are sources of gluons ($\neq \gamma$)

— g_s universal, fixed in gauge sector: color charges quantized

Lagrangian I of QCD:

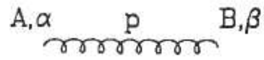
$$\mathcal{L} = \bar{q}(i\not{D} - m_q)q - \frac{1}{2}TrG^2_{\mu\nu}$$

$$= \bar{q}(i\not{\partial} - m_q)q - \frac{1}{2}Tr(\partial_\nu G_\mu - \partial_\mu G_\nu)^2 \quad \text{kinet. part}$$

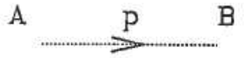
$$-g_s\bar{q}\not{G}q \quad q-g \text{ coupling}$$

$$+ig_sTr(\partial_\nu G_\mu - \partial_\mu G_\nu)[G_\mu, G_\nu] \quad 3g \text{ coupling}$$

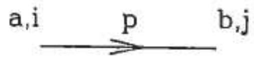
$$+\frac{g_s^2}{2}Tr[G_\mu, G_\nu]^2 \quad 4g \text{ coupling}$$



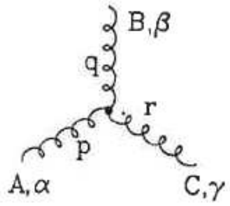
$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^\alpha p^\beta}{p^2+i\epsilon}\right]\frac{i}{p^2+i\epsilon}$$



$$\delta^{AB}\frac{i}{(p^2+i\epsilon)}$$

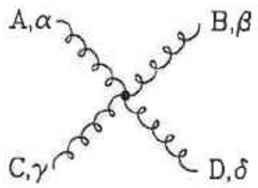


$$\delta^{ab}\frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

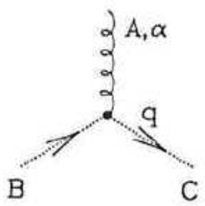


$$-g\ f^{ABC}[(p-q)^\gamma g^{\alpha\beta}+(q-r)^\alpha g^{\beta\gamma}+(r-p)^\beta g^{\gamma\alpha}]$$

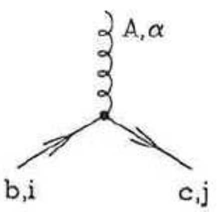
(all momenta incoming, $p+q+r = 0$)



$$-ig^2\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^2\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^2\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$



$$g\ f^{ABC}q^\alpha$$



$$-ig\ (t^A)_{cb}\ (\gamma^\alpha)_{ji}$$

§3. Feynman Path Integrals

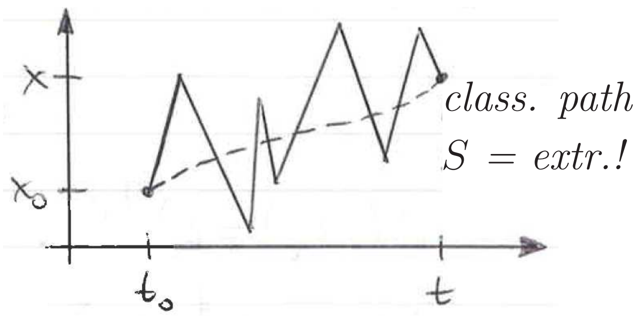
QM: transition ampl. of a particle $\{x_0, t_0\} \to \{x, t\}$:

$$\langle x,t|x_0,t_0\rangle = \langle x|e^{-iH(t-t_0)}|x_0\rangle = \langle x|e^{-iH\epsilon}e^{-iH\epsilon}\cdots|x_0\rangle$$
$$= \int\prod_i dx_i \langle x|e^{-iH\epsilon}|x_n\rangle\langle x_n|\cdots|x_0\rangle$$
$$\uparrow \mathbb{1} = \int dx_i |x_i\rangle\langle x_i|$$

$$\langle y_2|e^{-iH\epsilon}|y_1\rangle = \langle y_2|e^{-i\frac{\vec{p}^2}{2m}\epsilon}|y_1\rangle e^{-iV(y_1)\epsilon} \sim e^{i\mathcal{L}\epsilon}$$
$$\uparrow$$
$$\sim \int dk\; e^{-i\frac{k^2}{2m}\epsilon}\langle y_2|k\rangle\langle k|y_1\rangle \sim \int dk\; e^{-i\frac{k^2}{2m}\epsilon + i(y_2-y_1)k}$$
$$\sim \int dk\; \exp\left\{-i\frac{\epsilon}{2m}\left(k - m\frac{y_2-y_1}{\epsilon}\right)^2 + i\frac{m}{2\epsilon}(y_2-y_1)^2\right\}$$
$$\sim \exp\left\{i\frac{m}{2}\left(\frac{y_2-y_1}{\epsilon}\right)^2\epsilon\right\} = \exp\{iT_{kin}\epsilon\}$$

$$\boxed{\langle x,t|x_0,t_0\rangle \sim \int\mathcal{D}x\; \exp\, i\int dt\mathcal{L} \sim \int\mathcal{D}x\; e^{iS}}$$

transition amplitude =
sum over all histories
with weight e^{iS}



classical path = maximal weight due to extremal value of action S

Functionals: Mapping of numbers on functions

integral functional: $F(u) = \int dx' f(x', u(x'))$

derivative:

$$\frac{\delta F(u)}{\delta u(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dx' \left\{ f[x', u(x') + \epsilon\delta(x - x')] - f(x', u(x')) \right\}$$

$$= \left. \frac{\partial f}{\partial u} \right|_x$$

properties analogous to usual derivatives

Theorem: Green's functions are calculable as the derivative of the action functional (with external source)

Green's function:

$$G(x_1, \ldots, x_n) = {}_H\langle 0|T\{\phi_H(x_1) \cdots \phi_H(x_n)\}|0\rangle_H \quad \text{Heisenberg pic.}$$

$$= \langle 0|T\{\phi(x_1) \cdots \phi(x_n) S\}|0\rangle \quad \text{interaction pic. w/o vac. graphs}$$

determine S matrix

action functl.: $W(j) = \langle 0|T \exp i \int d^4x \{\mathcal{L}_{int}(\phi) + j\phi\}|0\rangle$ (int. pic.)

$$\Rightarrow i^n G(x_1, \ldots, x_n) = \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)} W(j)/W(0) \bigg|_{j=0}$$

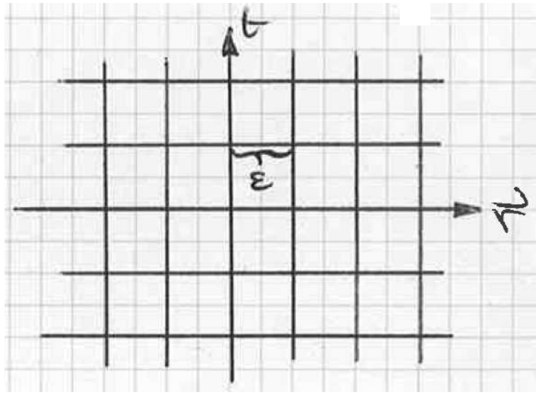
free action functional: $W(j) = \exp i \int d^4x \, \mathcal{L}_{int} \left(\frac{1}{i} \frac{\delta}{\delta j(x)} \right) W_0(j)$

$$W_0(j) = \langle 0|T \exp i \int d^4x \; j(x)\phi(x)|0\rangle$$

$$= \exp\{-\tfrac{i}{2} \int d^4x \; d^4y \; j(x)\Delta_F(x - y) j(y)\}$$

Path integral representation of field theory:

Minkowski space div. in cubes α with edge length $\epsilon \to 0$;



$\phi_\alpha = \frac{1}{\epsilon^4}\int d^4x\ \phi(x)$;

$\int d^4x\ f(x) = \sum \epsilon^4 f_\alpha$;

$\delta_4(x-y) = \frac{1}{\epsilon^4}\delta_{\alpha\beta}$

$$\left.\begin{aligned}\partial\phi(x) &= \int d^4y\ [-\partial_y\delta_4(x-y)]\phi(y) = \epsilon^4\partial_{\alpha\beta}\phi_\beta \\ &= \tfrac{1}{2\epsilon}[\phi_{\alpha+1}-\phi_{\alpha-1}] = \tfrac{1}{2\epsilon}[\delta_{\alpha+1,\beta}-\delta_{\alpha-1,\beta}]\phi_\beta\end{aligned}\right\} \begin{aligned}&\partial_{\alpha\beta} = \tfrac{1}{2\epsilon^5}[\delta_{\alpha+1,\beta}-\delta_{\alpha-1,\beta}] \\ &\text{skew symmetric}\end{aligned}$$

def. funct. $\mathcal{F}(\phi)$ by cont. integral:

$$\boxed{\begin{aligned}\mathcal{F}(\phi) &= \lim_{\epsilon\to 0}\int\prod_\alpha d\phi_\alpha F\left(\sum_\beta \epsilon^4 f(\phi_\beta)\right) \\ &= \int \mathcal{D}\phi\ F\left(\int d^4x\ f(\phi(x))\right)\end{aligned}}$$

representation of free action functional by a cont. integral

$$\tilde{W}_0(j) = \int \mathcal{D}\phi \exp i\int d^4x(\mathcal{L}_0 + j\phi) \qquad \mathcal{L}_0 = \tfrac{1}{2}\phi(-\partial^2 - m^2 + i\epsilon)\phi$$

free Lagrangian

$$= \int\prod_\alpha d\phi_\alpha \exp i\left[\epsilon^8\frac{1}{2}\phi_\alpha K_{\alpha\beta}\phi_\beta + \epsilon^4 j_\alpha\phi_\alpha\right]$$

trf. K to main diagonal: $K = V^T K' V \quad K' =$ diagonal

(K symm. $\Rightarrow V$ orthogonal) $\phi' = V\phi$

$$\prod_\alpha d\phi_\alpha = \prod_\alpha d\phi'_\alpha \quad [\text{Det } V = 1]$$

$$\tilde{W}_0(j) = \int\prod_\alpha d\phi'_\alpha \exp i\left[\epsilon^8\tfrac{1}{2}\phi'_\alpha K'_{\alpha\alpha}\phi'_\alpha + \epsilon^4(Vj)_\alpha\phi'_\alpha\right]$$

Fresnel integral: $\displaystyle\int_{-\infty}^{\infty} dx\ e^{i(\rho x^2+\sigma x)} = \sqrt{\frac{\pi i}{\rho}}e^{-i\frac{\sigma^2}{4\rho}}$

$$\tilde{W}_0(j) \sim \prod_\alpha \frac{1}{\sqrt{K'_{\alpha\alpha}}}e^{-\frac{i}{2}(Vj)^T_\alpha K'^{-1}_{\alpha\alpha}(Vj)_\alpha} \sim \exp\left\{-\tfrac{i}{2}j_\alpha K^{-1}_{\alpha\beta}j_\beta\right\} \sim W_0(j)$$

$\uparrow$ Det K' = Det K

<u>if:</u> $\frac{1}{\epsilon^8} K^{-1}_{\alpha\beta} = \Delta_F(x-y)$ Feynman prop. in position space

<u>Solution:</u>

$$K_{\alpha\gamma} K^{-1}_{\gamma\beta} = \delta_{\alpha\beta} \ \Rightarrow \ \epsilon^4 K_{\alpha\gamma} \times \frac{1}{\epsilon^8} K^{-1}_{\gamma\beta} = \frac{1}{\epsilon^4} \delta_{\alpha\beta}$$

$$(-\partial^2 - m^2 + i\epsilon)\Delta_F(x-y) = \delta_4(x-y) \Rightarrow \int d^4z \left\{(-\partial_x^2 - m^2 + i\epsilon)\delta_4(x-z)\right\}\Delta_F(z-y) = \delta_4(x-y)$$

comparison: $K_{\alpha\beta} = (-\partial^2 - m^2 + i\epsilon)\delta_4(x-y)$
Klein–Gordon operator

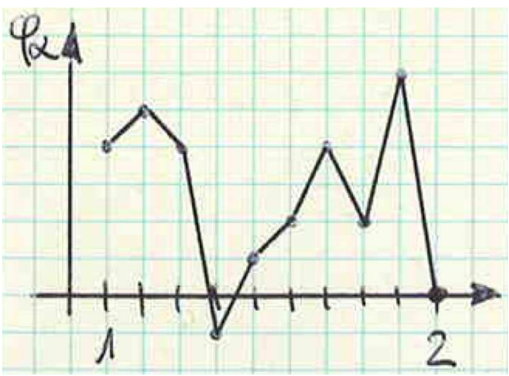
introduction of full Lagrangian by
$W(j) = \exp i \int d^4x \ \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta j(x)}\right) W_0(j)$

$$\begin{aligned} W(j) &\sim \int \mathcal{D}\phi \ \exp\left\{ i \int d^4x [\mathcal{L} + j\phi] \right\} \\ &\sim \int \mathcal{D}\phi \ \exp\left\{ i \left[S + \int d^4x \ j\phi \right] \right\} \end{aligned}$$

solution of QFT traced back to integration

propagator: $G(x_1, x_2) = \int \mathcal{D}\phi \ \phi(x_1)\phi(x_2) \exp i \int d^4x \mathcal{L} \Big/ \int \mathcal{D}\phi \ e^{iS}$

$$= \int d\phi_1 \ d\phi_2 \ \phi_1\phi_2 \int \prod_{\alpha \neq 1,2} d\phi_\alpha \exp i\epsilon^4 \sum_\beta \mathcal{L}(\phi_\beta) \Big/ \cdots$$



path integral:

1.) perturbatively solvable by expansion in coupling and successive Fresnel integration

2.) for strong coupling numerical integration of integrals that are defined on space-time-lattices

FERMIONS: incorporation of Pauli principle

Grassmann variables: η_i anti-comm. c-#'s $\{\eta_i, \eta_j\} = 0$

$$\eta_i^2 = 0$$

functions: polynomials $f(\eta) = a_0 + a_1\eta$

$$f(\eta_1, \eta_2) = a_0 + a_1\eta_1 + a_2\eta_2 + a_3\eta_1\eta_2$$

$$\vdots$$

differentiation: $\frac{\partial}{\partial\eta_i}\eta_j = \delta_{ij}$ $\qquad$ $\left\{\frac{\partial}{\partial\eta_i}, \eta_j\right\} = \delta_{ij}$

$$\left\{\frac{\partial}{\partial\eta_i}, \frac{\partial}{\partial\eta_j}\right\} = 0$$

integration:

$z = \int_{-\infty}^{\infty} dx\ f(x) = \int_{-\infty}^{\infty} dx\ f(x+a)$ for usual numbers

Grassmann variable: $z = \int d\eta\ f(\eta) = \int d\eta\ f(\eta + \zeta)$

$$= \int d\eta\ (f_0 + f_1\eta) = \int d\eta\ (f_0 + f_1\eta + f_1\zeta) \Rightarrow$$

$$\int d\eta = 0$$

$$\int d\eta\ \eta = 1$$

multi-dim.: $\{d\eta_i, d\eta_j\} = 0$

$$\{\eta_i, d\eta_j\} = 0$$

(a) integration $\equiv$ differentiation: $\int d\eta\, f(\eta) = \frac{\partial}{\partial\eta} f(\eta)$

(b) var. trf.: $I = \int d\eta_1 \cdots d\eta_n \; g(\eta) \quad g(\eta) = \cdots + g_1 \eta_1 \cdots \eta_n$

$= \pm g_1$

$\eta = M\zeta \Rightarrow \eta_1 \cdots \eta_n = \text{Det}\, M \;\; \zeta_1 \cdots \zeta_n : \;\; g(\eta) \sim \text{Det}\, M \;\; g(\zeta)$

[proof: $\eta_1\eta_2 = (M_{11}\zeta_1 + M_{12}\zeta_2)(M_{21}\zeta_1 + M_{22}\zeta_2)$

$= (M_{11}M_{22} - M_{12}M_{21})\zeta_1\zeta_2$]

$$\Rightarrow \underline{\underline{d\eta_1 \cdots d\eta_n = \text{Det}\, M^{-1} d\zeta_1 \cdots d\zeta_n}}$$

Grassmann fields: $\eta_i \to \eta(x)$ cont. with $\{\eta(x), \eta(y)\} = 0$

path integral:

$$W = \int \mathcal{D}\eta\, F\left(\int d^4x \;\, g\big(\eta(x)\big)\right) = \int \prod_x d\eta_x\, F\left[\epsilon^4 \sum_y g(\eta_y)\right]$$

Mathews–Salam formulae

$$\int \mathcal{D}\bar\eta \;\, \mathcal{D}\eta \;\, e^{-\bar\eta Q \eta} \sim \text{Det}\, Q$$

$$\int \mathcal{D}\bar\eta \;\, \mathcal{D}\eta \;\, \eta_i \eta_j \;\, e^{-\bar\eta Q \eta} \sim Q^{-1}_{ij} \;\, \text{Det}\, Q \qquad \text{etc.}$$

fermionic action functional:

$$\underline{\underline{W_{\eta\bar\eta} = \int \mathcal{D}\bar\psi \;\, \mathcal{D}\psi \;\, e^{i \int d^4x \;\, [\mathcal{L}(\bar\psi, \psi) + \bar\psi\eta + \bar\eta\psi]}}}$$

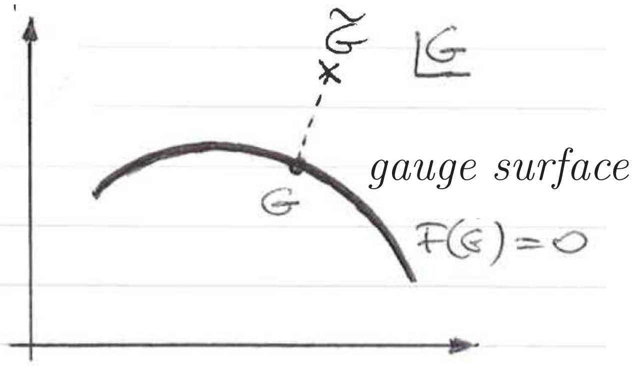
Green's functions by derivatives w.r.t. sources $\eta(x)$ and $\bar\eta(x)$

§4. Gauge Fixing

action functional: $W \sim \int \mathcal{D}G \ \exp \ i \int d^4x \ \mathcal{L}$

integration over infinite regions in which $\mathcal{L}$ does not change:

- rearrange integration such that integration regions of physically different and gauge-equivalent field configurations are separated
- factorize infinite volume: field configs only on gauge surface



gauge fixing: $F(G) = 0$

Note: For all $\tilde{G}$ there is a unique gauge trf. α such that

$\tilde{G} \underset{\alpha}{\rightarrow} G \qquad$ with $F(G) = 0$

Ex.: QED $\quad A_\mu = A'_\mu - \partial_\mu \Lambda \qquad F(A) = \partial_\mu A^\mu$

$$\partial_\mu A^\mu = 0 = \partial_\mu A'^\mu - \partial^2 \Lambda \Rightarrow \Lambda = \partial^{-2}(\partial A')$$

integration reordering: $\Delta(G) \int \mathcal{D}\tau_\alpha \ \delta\left(F(G^\alpha)\right) = 1$

in detail: $\Delta^{-1}(G) = \int \prod_x d\tau_{\alpha(x)} \prod_{x,a} \delta\left(F^a(G^\alpha_\mu)\right)$

with $d\tau_{\alpha(x)}$ = Hurwitz measure of $SU(3)$

$U(\alpha) = e^{-iT^a\alpha^a} : \quad U(\alpha')U(\alpha) = U(\alpha' \cdot \alpha)$

$\Rightarrow U(\alpha' \cdot \alpha) = e^{-iT^a(\alpha' \cdot \alpha)^a}$ with $(\alpha' \cdot \alpha)^a = \phi^a(\alpha', \alpha)$

hence: $d\tau_\alpha = J^{-1}(\alpha) d\alpha^1 \cdots d\alpha^8$ with $J(\alpha) = \mathrm{Det}\left[\left.\frac{\partial \phi^a}{\partial \alpha'_b}\right|_{\alpha'_b = 0}\right]$

fulfills: $d\tau_{\alpha' \cdot \alpha} = d\tau_{\alpha \cdot \alpha'} = d\tau_\alpha$ [gauge invariant]

$\Delta(G)=\Delta(G^\beta)$ is gauge invariant:

$$\Delta^{-1}(G^\beta)=\int\mathcal{D}\tau_\alpha\,\delta\left(F\left(G^{\beta\alpha}\right)\right)=\int\mathcal{D}\tau_{\beta\alpha}\,\delta\left(F\left(G^{\beta\alpha}\right)\right)=\Delta^{-1}(G)$$

action functional:

$$W\sim\int\mathcal{D}G\ \Delta(G)\int\mathcal{D}\tau_\alpha\ \delta\left(F\left(G^\alpha\right)\right)\ \ \exp i\int d^4x\ \mathcal{L}(G)$$

$$\underset{\text{gauge inv.}}{=}\int\mathcal{D}\tau_\alpha\int\mathcal{D}G^\alpha\ \Delta(G^\alpha)\,\delta\left(F\left(G^\alpha\right)\right)\exp i\int d^4x\ \mathcal{L}\left(G^\alpha\right)$$

$\uparrow SU(3)_\mathcal{C}$ invariant

$$=\int\mathcal{D}\tau_\alpha\times\int\mathcal{D}G\ \Delta\left(G\right)\delta\left(F\left(G\right)\right)\exp i\int d^4x\ \mathcal{L}\left(G\right)$$

$$\sim\int\mathcal{D}G\ \Delta\left(G\right)\delta\left(F\left(G\right)\right)\exp i\int d^4x\ \mathcal{L}\left(G\right)$$

integration over gauge surface and gauge orbit orthogonalized: $\int\mathcal{D}\tau_\alpha\sim$ const. factorized

infinitesimal: $F\left(G^\alpha(x)\right)=F\left(G(x)\right)+\int d^4y\ M_F(x,y)\ \alpha(y)+\cdots$

$$\Delta^{-1}(G)=\int\mathcal{D}\tau_\alpha\ \delta(M_F\alpha)\sim\int\mathcal{D}\alpha\ \delta(M_F\alpha)\sim\text{Det}\,M_F^{-1}$$

$\Delta(G)=\text{Det}\,M_F$ Faddeev–Popov determinant

$$M_F=\left.\frac{\delta F\left(G^\alpha\right)}{\delta\alpha}\right|_{\alpha=0}$$

EXAMPLES: $\left(G^\alpha\right)^a_\mu=G^a_\mu-f_{abc}G^b_\mu\alpha^c+\frac{1}{g_s}\partial_\mu\alpha^a+\mathcal{O}(\alpha^2)$

(i) Lorenz gauge: $\partial G=f$

$$\partial^\mu\left(G^\alpha\right)^a_\mu-f^a=\underbrace{\left(\partial^\mu G^a_\mu-f^a\right)}_{=0}\ \underbrace{-f_{abc}\partial^\mu G^b_\mu\alpha^c+\frac{1}{g_s}\partial^2\delta_{ab}\alpha^b}_{\frac{1}{g_s}\int d^4y\ \left\{\partial^2\delta_{ab}+g_sf_{abc}\partial^\mu G^c_\mu(x)\right\}\delta_4(x-y)\alpha^b(y)}$$

$$M^{ab}_L(x,y)=\frac{1}{g_s}\left[\partial^2\delta_{ab}+g_sf_{abc}\partial^\mu G^c_\mu\right]\delta_4(x-y)$$

non-Abelian: $\text{Det}\,M_L$ manifestly gauge field-dependent

Abelian, QED: $\text{Det}\,M_L$ independent of $A\rightarrow$ ineff.

(ii) Axial gauge: $nG = 0$ $\quad n^2 = \pm 1, 0$
[temporal/axial/lightcone]

$$n\,(G^\alpha)^a = \underbrace{nG^a}_{=0} - f_{abc}\underbrace{nG^b}_{=0}\alpha^c + \frac{1}{g_s} n\partial\alpha^a$$
$$= \frac{1}{g_s}\int d^4y\ \delta_{ab}\ n\partial\ \delta_4(x-y)\ \alpha^b(y)$$

$$\underline{\underline{M_A^{ab}(x,y) = \frac{1}{g_s}\ n\partial\ \delta_{ab}\delta_4(x-y)}}$$

independent of G $\rightarrow$ ineffective

Effective Lagrangian: $W \sim \int \mathcal{D}\text{fields}\ \exp i\int d^4x\ \mathcal{L}_{eff}$

Lorenz gauge: phys. results gauge invariant and independent of f

$\Rightarrow$ average over all f with weight ρ $\Big\}$ $\rho(f) = \exp\frac{-i}{\xi}\int d^4x\ Trf^2$
$\underline{\underline{\xi = \text{free gauge parameter}}}$

$$W \sim \int \mathcal{D}f\ \rho(f) \int \mathcal{D}G\ \delta(\partial G - f) \text{Det}\, M_L \exp\ i\int d^4x\ \mathcal{L}$$
$$\sim \int \mathcal{D}G\ \text{Det}\, M_L \exp\ i\int d^4x\ (\mathcal{L} + \mathcal{L}_{GF})$$

$$\underline{\underline{\mathcal{L}_{GF} = -\frac{1}{\xi} Tr(\partial G)^2 \quad \text{gauge fixing term}}}$$

ghosts: Grassmann octet-fields

$$\text{Det}\, M_L \sim \int \mathcal{D}\tilde{c}^*\mathcal{D}\tilde{c}\ \exp -i\int d^4x\ d^4y\ \tilde{c}_a^*(x) M_L^{ab}\tilde{c}_b(y)$$
$$\underset{(\tilde{c}=\sqrt{g_s}c)}{\sim} \int \mathcal{D}c^*\mathcal{D}c\ \exp i\int d^4x\ \left\{(\partial^\mu c_a^*)(\partial_\mu c_a) + g_s f_{abc}(\partial^\mu c_a^*) G_\mu^c c_b\right\}$$
$$\sim \int \mathcal{D}c^*\mathcal{D}c\ \exp i\int d^4x\ \mathcal{L}_{FPG}$$

$\uparrow$ ghost Lagrangian $= \partial c^* Dc$

ghost fields: fermionic spinless auxiliary fields $\rightarrow$ non existing in reality [spin-statistics theorem]; contribute only in loops coupled to gluons.

Full Lagrangian of QCD:

$$W \sim \int \mathcal{D}\bar{q}\ \mathcal{D}q\ \mathcal{D}G\ \mathcal{D}c^*\ \mathcal{D}c\ \exp i \int d^4x\ \mathcal{L}_{eff}$$

$\mathcal{L}_{eff} = \mathcal{L}_{QCD} + \mathcal{L}_{GF} + \mathcal{L}_{FPG}$:

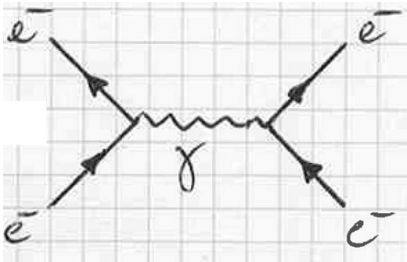
$$\mathcal{L}_{QCD} = qg \text{ Lagrangian}$$
$$\mathcal{L}_{GF} = \text{gauge fixing}$$
$$\mathcal{L}_{FPG} = \text{ghost Lagrangian}$$

Lorenz gauge:	axial gauge:
$\mathcal{L}_{GF} = -\frac{1}{\xi} Tr(\partial G)^2$	$\mathcal{L}_{GF} = -\frac{1}{\xi} Tr(nG)^2$ for $\xi \rightarrow 0$
$\mathcal{L}_{FPG} = \partial c^* (\partial + g_s f G) c$	$\mathcal{L}_{FPG} = 0$

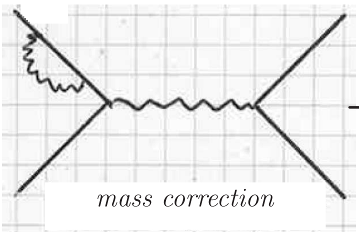
§5. Asymptotic Freedom

REM. QED: $e^- e^-$ scattering: $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 + \cdots$

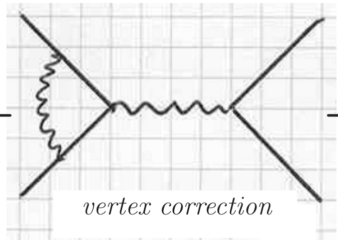
Bornterm:

$\mathcal{M}_0 =$  $= \dfrac{4\pi\alpha(\mu^2)}{Q^2} \cdots \quad [Q^2 = -q^2]$

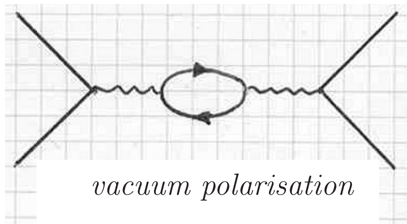
Radiative corrections:

$\mathcal{M}_1 =$ 

mass correction

$+$ 

vertex correction

$+$ 

vacuum polarisation

$+$ C.T.

Elements [asymptotic]:

1.) electron propagator: $G(p) = \dfrac{i}{\not p}\left\{1 + \xi \dfrac{\alpha}{4\pi} e_f^2 \log\left(\dfrac{-p^2}{\mu^2}\right)\right\}$

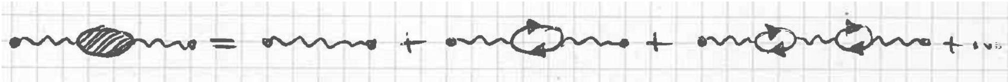
2.) vertex: $\Gamma_\mu = -iee_f\gamma_\mu \left\{1 - \xi \dfrac{\alpha}{4\pi} e_f^2 \log \dfrac{Q^2}{\mu^2} + \cdots \right\}$

↑ IR part

3.) photon prop.: $\Pi_{\mu\nu} = -i\dfrac{\alpha}{3\pi}\sum_f e_f^2 \left[q_\mu q_\nu - q^2 g_{\mu\nu}\right] \log \dfrac{Q^2}{\mu^2}$

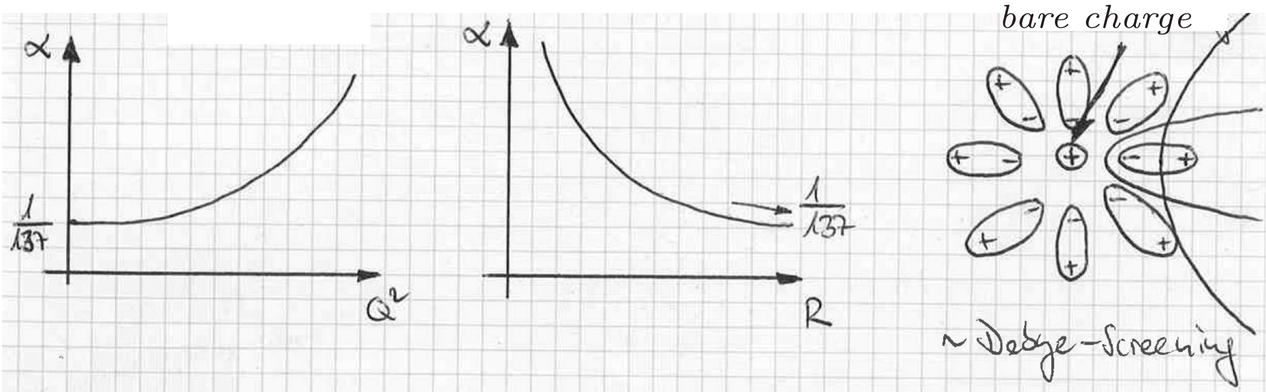
$$\Rightarrow \mathcal{M} = \mathcal{M}_0 \left[1 + \tfrac{\alpha}{3\pi}\textstyle\sum_f e_f^2 \log \dfrac{Q^2}{\mu^2} + \cdots \right] \equiv \dfrac{4\pi\alpha(Q^2)}{Q^2}\cdots$$

$$\Rightarrow \alpha(Q^2) = \alpha(\mu^2)\left[1 + \tfrac{\alpha}{3\pi}\textstyle\sum_f e_f^2 \log \dfrac{Q^2}{\mu^2}\right]$$

Summation: 

effective electric charge:
$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha}{3\pi}\sum_f e_f^2 \log\frac{Q^2}{\mu^2}}$$

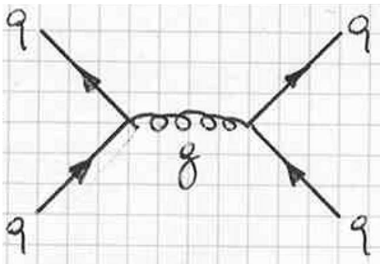
Screening effect:



Translation to QCD: quark-quark scattering

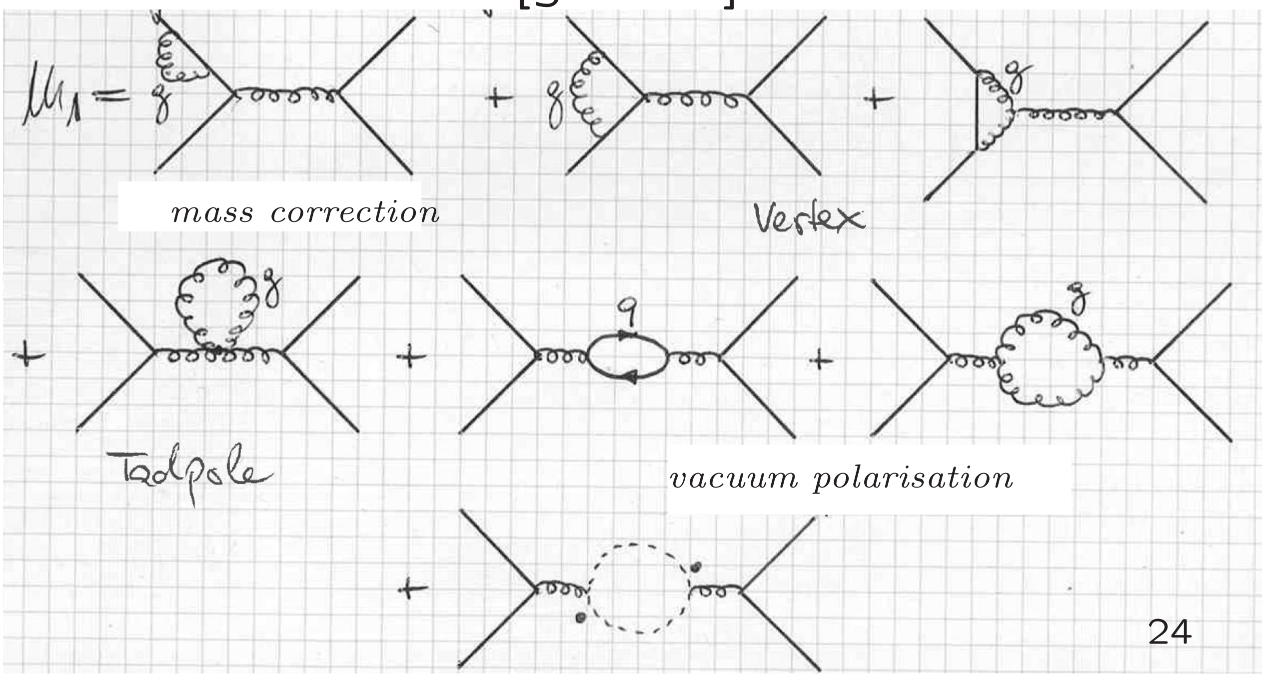
$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 + \cdots$

Bornterm:

$\mathcal{M}_0 =$ 

$= \frac{4\pi\alpha_s(\mu^2)}{Q^2}\cdots$

Radiative corrections [generic]:



Elements [asymptotic]:

1.) quark propagator: $e^2 \to g_s^2 (T^a T^a)_{ij} = \frac{N^2-1}{2N} g_s^2 \delta_{ij}$

$$G_{ij}(p) = i \frac{\delta_{ij}}{\not p} \left\{ 1 + \xi \frac{\alpha_s}{4\pi} \frac{N^2-1}{2N} \log\left(\frac{-p^2}{\mu^2}\right) \right\}$$

2.) vertex:

$$\Gamma^a_{\mu,ij} = -i T^a_{ij} g_s \gamma_\mu \left\{ 1 - \frac{\alpha_s}{4\pi} \log \frac{Q^2}{\mu^2} \left[\xi \frac{N^2-1}{2N} + \left(1 - \frac{1-\xi}{4}\right) N \right] + (IR) \right\}$$

3.) gluon propagator: tadpole = 0

fermion loop: $e^2 e_f^2 \to g_s^2 Tr(T^a T^b) = \frac{1}{2} g_s^2 \delta^{ab}$

$$I^q_{\mu\nu} = -i \frac{\alpha_s}{3\pi} N_F \left[q_\mu q_\nu - q^2 g_{\mu\nu} \right] \frac{\delta^{ab}}{2} \log \frac{Q^2}{\mu^2}$$

gluon loop:

$$I^g_{\mu\nu} = i \frac{\alpha_s}{4\pi} N \delta^{ab} \left[\frac{11}{6} q_\mu q_\nu - \frac{19}{12} q^2 g_{\mu\nu} + \frac{1-\xi}{2} \left(q_\mu q_\nu - q^2 g_{\mu\nu} \right) \right] \log \frac{Q^2}{\mu^2}$$

not transverse / gauge dependent

ghost loop:

$$I^G_{\mu\nu} = -i \frac{\alpha_s}{4\pi} N \delta^{ab} \left[\frac{1}{6} q_\mu q_\nu + \frac{1}{12} q^2 g_{\mu\nu} \right] \log \frac{Q^2}{\mu^2}$$

$$\Rightarrow I^g_{\mu\nu} + I^G_{\mu\nu} = i \frac{\alpha_s}{4\pi} N \delta^{ab} \left(\frac{5}{3} + \frac{1-\xi}{2} \right) \left(q_\mu q_\nu - q^2 g_{\mu\nu} \right) \log \frac{Q^2}{\mu^2}$$

ghosts transversalize g loop, but gauge dep.

$$i \frac{-g_{\mu\nu} + (1-\xi) \frac{q_\mu q_\nu}{q^2}}{q^2} \to i \frac{-g_{\mu\rho} + (1-\xi) \frac{q_\mu q_\rho}{q^2}}{q^2} I^{\rho\sigma} i \frac{-g_{\sigma\nu} + (1-\xi) \frac{q_\sigma q_\nu}{q^2}}{q^2}$$

Sum of all terms:

$$\mathcal{M}(qq \to qq) = \mathcal{M}_0 \left\{ 1 + \frac{\alpha_s}{4\pi} \left[\frac{2}{3} N_F - \frac{13}{6} N - \frac{3}{2} N \right] \log \frac{Q^2}{\mu^2} + (IR) \right\}$$

$\uparrow$ q-loop, $\uparrow$ g-loop, $\uparrow$ g-vertex

$$\equiv \frac{4\pi\alpha_s(Q^2)}{Q^2} \cdots$$

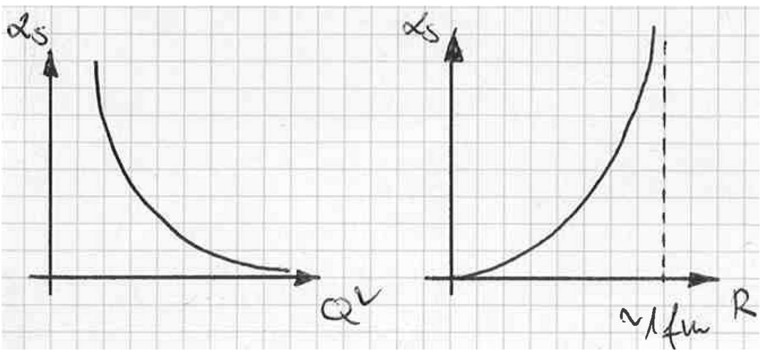
$$\Rightarrow \alpha_s(Q^2) = \alpha_s(\mu^2) \left[1 - \frac{11N - 2N_F}{12} \frac{\alpha_s}{\pi} \log \frac{Q^2}{\mu^2} \right] + \mathcal{O}(\alpha_s^3)$$

Summation:

$$\boxed{\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{33-2N_F}{12} \frac{\alpha_s}{\pi} \log \frac{Q^2}{\mu^2}}}$$

With increasing Q^2 the effective color charge vanishes: asymptotic freedom

[non-Abelian $SU(3)$: $N_F \leq 16$] consequence of non-Abelian gauge boson loops; contrary to $U(1)$ [and all other theories] [Politzer '73, Gross & Wilczek '73; 't Hooft ?]



$\sim$ confinement radius

Scale parameter of QCD: quantum theory introduces a scale into unscaled classical chromodynamics [for $m_q = 0$] via renormalization: introduction of coupling constant at default distance:

$\alpha_s = \alpha_s(\mu^2)$ [$\leftarrow$ exp. determined]

Reformulation:

$$\frac{1}{\alpha_s(Q^2)} = \underbrace{\frac{1}{\alpha_s(\mu^2)} - \frac{33-2N_F}{12\pi}\log\mu^2}_{\equiv \frac{33-2N_F}{12\pi}\log\frac{1}{\Lambda^2}} + \frac{33-2N_F}{12\pi}\log Q^2$$

$$\Rightarrow \boxed{\alpha_s(Q^2) = \frac{12\pi}{(33-2N_F)\log\frac{Q^2}{\Lambda^2}}}$$

$\Lambda^{-1} \sim 1$ fm $\sim$ conf. rad.

$\Rightarrow$ $\boxed{\Lambda \sim 100 - 300 \text{ MeV}}$

$\frac{\alpha_s(Q^2)}{\pi} \lesssim 10^{-1}$ for $Q^2 \gtrsim 2\text{GeV}^2$

$\Rightarrow$ region of ensured perturbation theory

Renormalization group equation:

$$\mu^2 \frac{\partial\alpha_s(\mu^2)}{\partial\mu^2} = \beta\left(\alpha_s(\mu^2)\right) \qquad \beta(\alpha_s) = -\beta_0\frac{\alpha_s^2}{\pi} + \mathcal{O}(\alpha_s^3)$$

Solution: $\log\frac{Q^2}{\mu^2} = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{\beta(\alpha_s)} = -\frac{\pi}{\beta_0}\left[\frac{1}{\alpha_s(\mu^2)} - \frac{1}{\alpha_s(Q^2)}\right]$

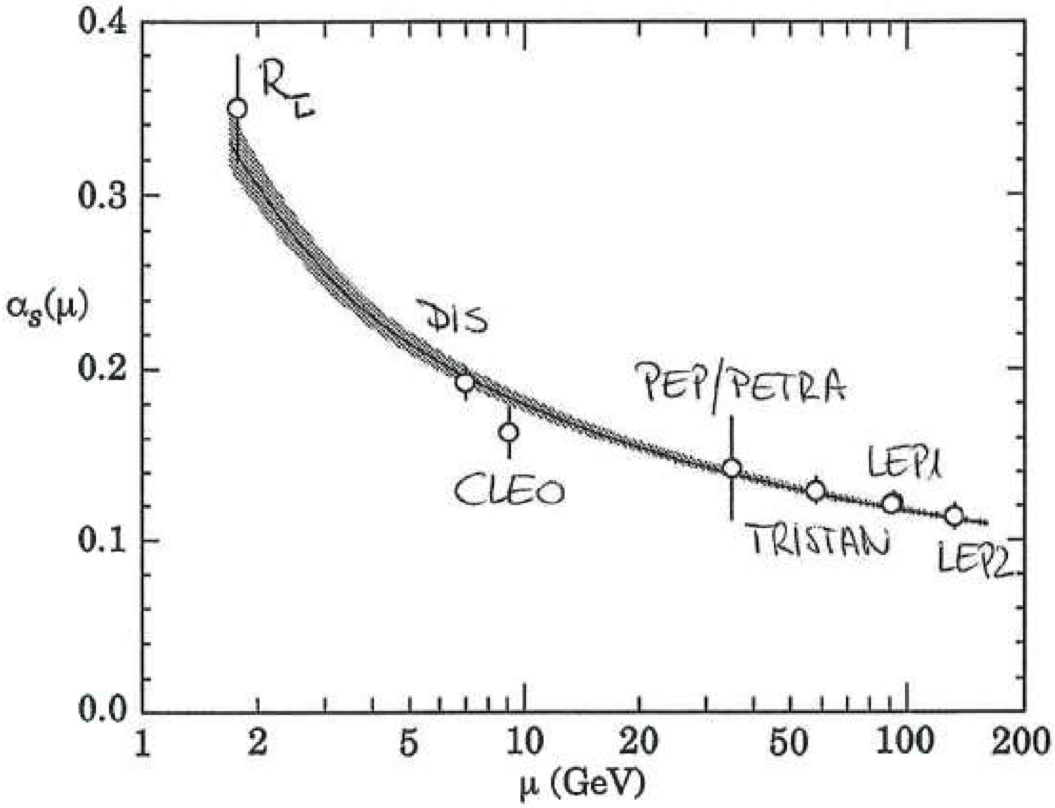
$$\boxed{\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+\beta_0\frac{\alpha_s}{\pi}\log\frac{Q^2}{\mu^2}} \qquad \beta_0 = \frac{33-2N_F}{12}}$$

RGG det. asymptotic behavior of running cplg.

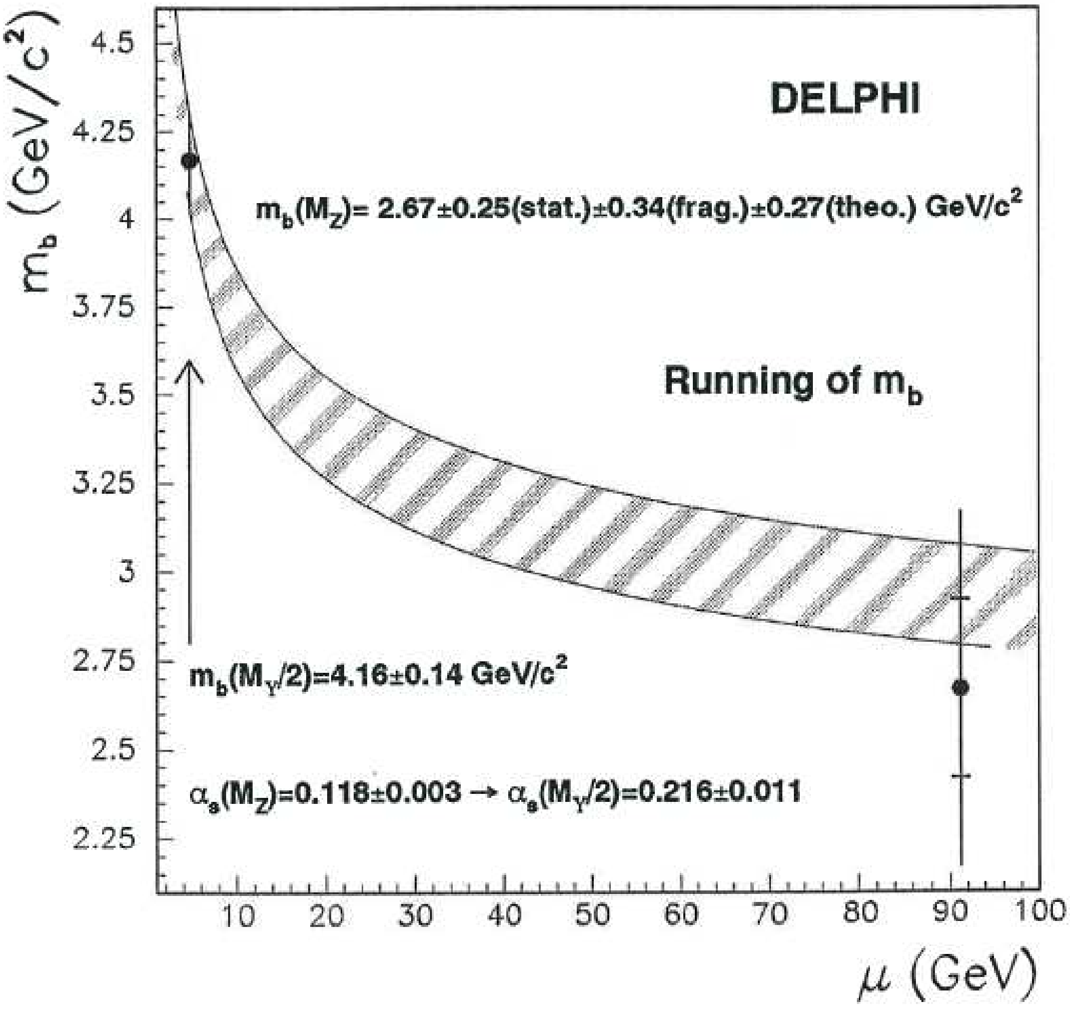
higher orders: $\beta(\alpha_s) = -\frac{\alpha_s^2}{\pi}\left[\beta_0 + \beta_1\frac{\alpha_s}{\pi} + \beta_2\frac{\alpha_s^2}{\pi^2} + \cdots\right]$

$$\beta_1 = \frac{153-19N_F}{24} \qquad \beta_2 = \frac{1}{128}\left[2857 - \frac{5033}{9}N_F + \frac{325}{27}N_F^2\right]$$

$$\alpha_s(Q^2) = \frac{\pi}{\beta_0\log\frac{Q^2}{\Lambda^2}}\left\{1 - \frac{\beta_1}{\beta_0^2}\,\frac{\log\log\frac{Q^2}{\Lambda^2}}{\log\frac{Q^2}{\Lambda^2}} + \cdots\right\}$$



0.4
0.3
0.2
0.1
0.0
$\alpha_s(\mu)$
1
2
5
10
20
50
100
200
μ (GeV)
DIS
CLEO
PEP/PETRA
TRISTAN
LEP1
LEP2



m_b (GeV/c^2)
DELPHI
$m_b(M_Z)$= 2.67±0.25(stat.)±0.34(frag.)±0.27(theo.) GeV/c^2
Running of m_b
$m_b(M_\Upsilon/2)$=4.16±0.14 GeV/c^2
$\alpha_s(M_Z)$=0.118±0.003 → $\alpha_s(M_\Upsilon/2)$=0.216±0.011
4.5
4.25
4
3.75
3.5
3.25
3
2.75
2.5
2.25
10
20
30
40
50
60
70
80
90
100
μ (GeV)

RENORMALIZATION SCHEMES $[n = 4 - 2\epsilon]$

fermion propagator: $S^{-1}(p) = \not{p}[1 - \tilde{\Sigma}(p)]$

$$\tilde{\Sigma}(p) = \frac{4}{3}\frac{g_s^2}{(4\pi)^{2-\epsilon}}(\mu f)^{2\epsilon}\frac{\Gamma(\epsilon)}{(-p^2)^\epsilon}2(1-\epsilon)B(2-\epsilon, 1-\epsilon)$$

$$g_s^2 \to g_s^2(\mu f)^{2\epsilon}$$

$(\mu f)^{2\epsilon}$ $[f =$ arb. cons.$]$ makes action dimensionless!

$$S^{-1}(p) = \not{p}\left\{1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\left[\frac{1}{\epsilon} - \log\frac{-p^2}{(\mu f)^2} + 1 + \log(4\pi) - \gamma_E\right]\right\}$$

Euler constant ↑

$[\Gamma(x) = \frac{1}{x} - \gamma_E + \mathcal{O}(x)]$

multiplicative renormalization: $S^{-1}(p) = Z_\psi^{-1} S_R^{-1}(p)$

(i) Dyson's renormalization scheme

require: $f = 1$
$S_R^{-1}(p) = \not{p}$ for $\mu^2 = -p^2$ $\Big\}$ $S^{-1}(p) = \not{p}[1 - \tilde{\Sigma}(\mu)][1 - \tilde{\Sigma}(p) + \tilde{\Sigma}(\mu)]$

Solution: $Z_\psi^{-1} = 1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\left[\frac{1}{\epsilon} + \log(4\pi) - \gamma_E + 1\right]$

$$S_R^{-1}(p) = \not{p}\left[1 + \frac{4}{3}\frac{g_{sMOM}^2}{16\pi^2}\log\left(\frac{-p^2}{\mu^2}\right)\right]$$

(MOM = momentum subtraction)

Cplg./charge depends on renormalization scheme

(ii) 't Hooft: Minimal subtraction (MS)

require: $f = 1$

Z_ψ^{-1} removes only $\frac{1}{\epsilon}$ pole

$$S^{-1}(p) = \not{p}\left[1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\frac{1}{\epsilon}\right]\left\{1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\left[-\log\left(\frac{-p^2}{\mu^2}\right) + \log(4\pi) - \gamma_E + 1\right]\right\}$$

Solution: $Z_\psi^{-1} = 1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\frac{1}{\epsilon}$

$$S_R^{-1}(p) = \not{p}\left\{1 - \frac{4}{3}\frac{g_{sMS}^2}{16\pi^2}\left[-\log\left(\frac{-p^2}{\mu^2}\right) + \log(4\pi) - \gamma_E + 1\right]\right\}$$

(iii) Modified minimal subtraction ($\overline{\text{MS}}$) Bardeen,...

require: $f = \exp\left[-\frac{1}{2}(\log(4\pi) - \gamma_E)\right]$ ← rem. trivial constants

$$S^{-1}(p) = \not{p}\left[1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\frac{1}{\epsilon}\right]\left\{1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\left[1 - \log\left(\frac{-p^2}{\mu^2}\right)\right]\right\}$$

Solution: $Z_\psi^{-1} = 1 - \frac{4}{3}\frac{g_s^2}{16\pi^2}\frac{1}{\epsilon}$

$$S_R^{-1}(p) = \not{p}\left\{1 - \frac{4}{3}\frac{g_{s\overline{\text{MS}}}^2}{16\pi^2}\left[1 - \log\left(\frac{-p^2}{\mu^2}\right)\right]\right\}$$

MS $\leftrightarrow$ $\overline{\text{MS}}$: $\mu^2 \leftrightarrow \mu^2 \exp\left[-\log(4\pi) + \gamma_E\right]$

$$\Lambda_{MS}^2 = \mu^2 \exp\left\{-\frac{16\pi^2}{\beta_0\ g_{sMS}^2} + \frac{\beta_1}{\beta_0^2}\log(\beta_0\ g_{sMS}^2)\right\}$$

$$\Lambda_{\overline{\text{MS}}}^2 = \mu^2 \exp\left\{-\frac{16\pi^2}{\beta_0\ g_{s\overline{\text{MS}}}^2} + \frac{\beta_1}{\beta_0^2}\log(\beta_0\ g_{s\overline{\text{MS}}}^2)\right\}$$

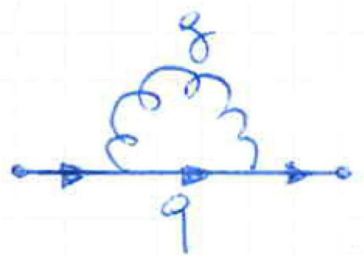
$$\boxed{\Lambda_{\overline{\text{MS}}} = \Lambda_{MS}\exp\left\{\frac{\log(4\pi) - \gamma_E}{2}\right\}}$$

β_0, β_1 independent of ren. scheme (not $\beta_{i\geq 2}$)

$\alpha_{s\overline{\text{MS}}}(Q^2) > \alpha_{sMS}(Q^2)$

Quark masses

quark self-energy:



$$\Sigma(\not p = m) = m\ C_F\ \frac{\alpha_s}{\pi}\Gamma(1+\epsilon)\left(\frac{4\pi\mu_0^2}{m^2}\right)^{\epsilon}\left(\frac{3}{4\epsilon}+1\right)$$

$m = m_0 + \Sigma(\not p = m)$ pole mass

$\overline{m}(\mu^2) = m_0 + \delta\overline{m}$ $\overline{\text{MS}}$ mass

$$\delta\overline{m} = m\, C_F \frac{\alpha_s}{\pi}\Gamma(1+\epsilon)\left(\frac{4\pi\mu_0^2}{\mu^2}\right)^{\epsilon}\frac{3}{4\epsilon}\quad \text{[only divergence]}$$

Relation pole mass $\leftrightarrow$ $\overline{\text{MS}}$ mass:

$$\begin{aligned}\overline{m}(\mu^2) &= m - [\Sigma(\not p = m) - \delta\overline{m}] = m\left[1 - C_F\frac{\alpha_s}{\pi}\left(\frac{3}{4}\log\frac{\mu^2}{m^2}+1\right)\right]\\ &= m\left[1 - C_F\frac{\alpha_s}{\pi}\right]\left[1-\frac{3}{4}C_F\frac{\alpha_s}{\pi}\log\frac{\mu^2}{m^2}\right]\end{aligned}$$

$$\boxed{\begin{aligned}\overline{m}(m^2) &= m\left[1 - C_F\frac{\alpha_s(m^2)}{\pi}\right]\\ \overline{m}(\mu^2) &= \overline{m}(m^2)\left[1 - \frac{\alpha_s}{\pi}\log\frac{\mu^2}{m^2}\right]\end{aligned}}$$

renormalization group equation:

$$\mu^2\frac{\partial\overline{m}(\mu^2)}{\partial\mu^2} = -\gamma_m(\alpha_s(\mu^2))\ \overline{m}(\mu^2)$$

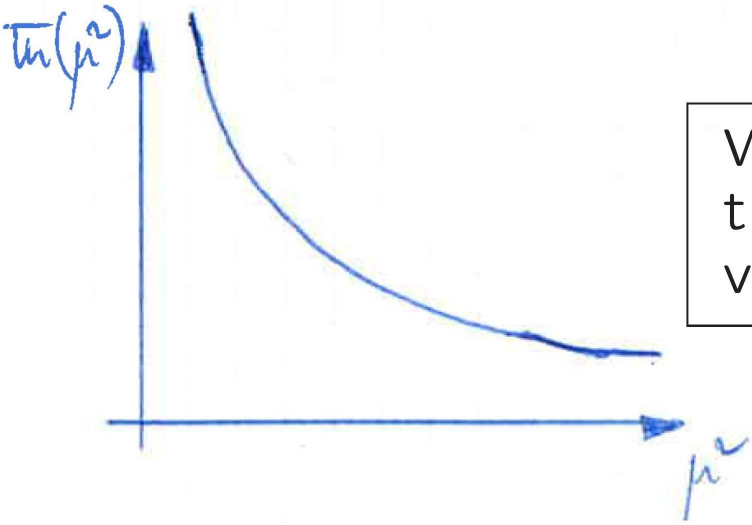
$\gamma_m(\alpha_s) = \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)$ anomalous mass dimension

$$\alpha_s(\mu^2) = \frac{\pi}{\beta_0 \log \frac{\mu^2}{\Lambda^2}}$$

$$\Rightarrow \text{ solution: } \overline{m}(\mu^2) = \overline{m}(m^2) \exp\left\{-\frac{1}{\beta_0}\int_{m^2}^{\mu^2} \frac{dQ^2}{Q^2 \log \frac{Q^2}{\Lambda^2}}\right\}$$

$$= \overline{m}(m^2)\left[\frac{\alpha_s(\mu^2)}{\alpha_s(m^2)}\right]^{\frac{1}{\beta_0}}$$

$$\boxed{\begin{aligned} \overline{m}(\mu^2) &= \hat{m}[\alpha_s(\mu^2)]^{\frac{1}{\beta_0}} \\ \hat{m} &= \overline{m}(m^2)[\alpha_s(m^2)]^{-\frac{1}{\beta_0}} \end{aligned}}$$ [RG-invariant]



With growing μ^2 $(R \to 0)$ the effective quark mass vanishes.

Examples:

bottom quark: $m_b = 4.8$ GeV $\quad \overline{m}_b(m_b^2) = 4.2$ GeV

$\overline{m}_b(M_Z^2) = 2.9$ GeV

charm quark: $m_c = 1.6$ GeV $\quad \overline{m}_c(m_c^2) = 1.2$ GeV

$\overline{m}_c(M_Z^2) = 0.6$ GeV

light quarks: $\overline{m}_u(1\,\text{GeV}^2) \sim 5$ MeV Gasser, Leutwyler

[QCD sum rules] $\overline{m}_d(1\,\text{GeV}^2) \sim 8$ MeV

$\overline{m}_s(1\,\text{GeV}^2) \sim 200$ MeV

Higher orders:

$$\overline{m}(m^2) = \frac{m}{1 + C_F \frac{\alpha_s(m^2)}{\pi} + K \left(\frac{\alpha_s(m^2)}{\pi}\right)^2}$$

Gray, Broadhurst, Grafe, Schilcher

$K_t \sim 10.9 \qquad K_b \sim 12.4 \qquad K_c \sim 13.4$

$$\overline{m}(\mu^2) = \overline{m}(m^2) \frac{c\left[\frac{\alpha_s(\mu^2)}{\pi}\right]}{c\left[\frac{\alpha_s(m^2)}{\pi}\right]}$$

$$\begin{aligned}
c(x) &= \left(\frac{9}{2}x\right)^{\frac{4}{9}} \left[1 + 0.895x + 1.371x^2 + 1.952x^3\right] \quad m_s < \mu < m_c \\
c(x) &= \left(\frac{25}{6}x\right)^{\frac{12}{25}} \left[1 + 1.014x + 1.389x^2 + 1.091x^3\right] \quad m_c < \mu < m_b \\
c(x) &= \left(\frac{23}{6}x\right)^{\frac{12}{23}} \left[1 + 1.175x + 1.501x^2 + 0.1725x^3\right] \quad m_b < \mu < m_t \\
c(x) &= \left(\frac{7}{2}x\right)^{\frac{4}{7}} \left[1 + 1.389x + 1.793x^2 - 0.6834x^3\right] \quad m_t < \mu
\end{aligned}$$

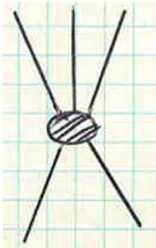
Chetyrkin

Larin, van Ritbergen, Vermaseren

§6. Renormalization Group

Parameters of a field theory [masses, couplings] are introduced for a certain μ^2; physical observables are independent of the particular choice of μ^2:
mod. of $\mu^2 \oplus$ corresponding change of parameters
$\Rightarrow$ invariance, formulated as RGEs [$\leftarrow$ partial DEs]
application: μ^2-variation moved to Q^2-variation by means of dimensional analysis
$\Rightarrow$ Q^2-variation of observables determined

Derivation of RGE:

Green's function: $G^{N_G N_\psi}(p) =$  $N_G = \#$ gauge fields, $N_\psi = \#$ fermion fields

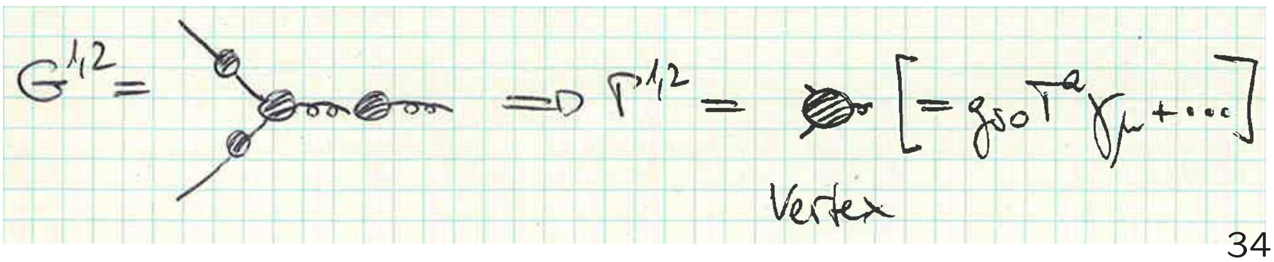
$$= \langle 0 | T\{\psi(x_1)\cdots\} | 0\rangle_{FT}$$

amputated Green's functions:

$$\Gamma^{N_G N_\psi}(p) = \frac{G^{N_G N_\psi}(p)}{\prod_G G^{2,0}(p_G) \prod_\psi G^{0,2}(p_\psi)}$$

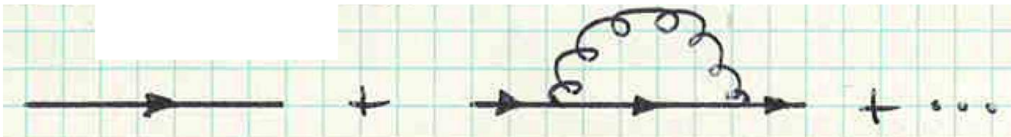
Examples:

$\Gamma^{0,2} = [G^{0,2}]^{-1}$



Divergent parts of Γ's can be separated as cut-off dependent factors; the remaining rest Γ_R is finite after the introduction of the renormalized coupling g and well-defined for cut-off $\to \infty$; the renormalization constants depend only on the species of the external legs.

Examples: (i) Fermion Propagator:

$iS'_F(p) =$ 

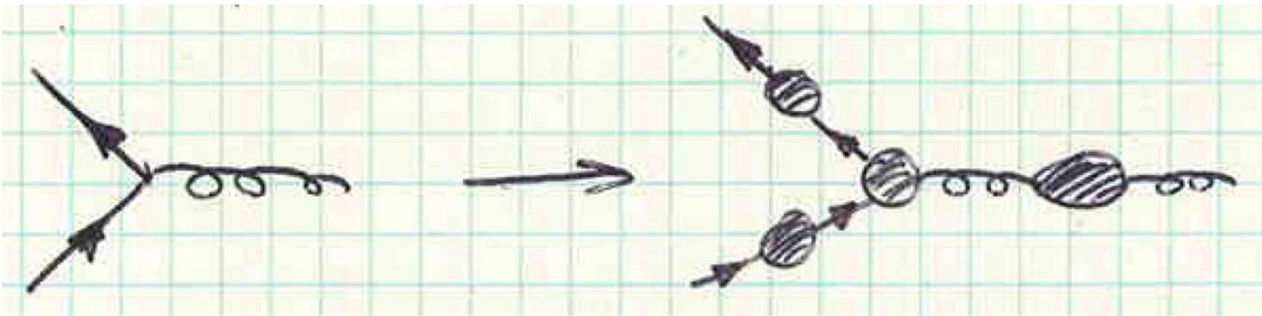
$$= \frac{i}{\not p} + \frac{i}{\not p}\left[-i\Sigma(\not p)\right]\frac{i}{\not p} + \cdots$$

$$\Sigma(\not p, \epsilon) = -C_F \frac{\alpha_s}{4\pi}\Gamma(1+\epsilon)\left(\frac{4\pi\mu^2}{-p^2}\right)^{\epsilon}\left(\frac{1}{\epsilon}+1+\mathcal{O}(\epsilon)\right)\not p$$

$$\begin{aligned}\frac{i}{\not p} &\to \frac{i}{\not p}\left[1 - C_F\frac{\alpha_s}{4\pi}\Gamma(1+\epsilon)\,(4\pi)^{\epsilon}\left(\frac{1}{\epsilon}+1\right) - C_F\frac{\alpha_s}{4\pi}\log\frac{\mu^2}{-p^2}\right] + \mathcal{O}(\epsilon)\\ &= \frac{i}{\not p}\left[1 - C_F\frac{\alpha_s}{4\pi}\Gamma(1+\epsilon)\,(4\pi)^{\epsilon}\left(\frac{1}{\epsilon}+1\right)\right]\left[1 - C_F\frac{\alpha_s}{4\pi}\log\frac{\mu^2}{-p^2}\right] + \mathcal{O}(\epsilon, \alpha_s^2)\end{aligned}$$

$$\begin{aligned} S'_F(p) &= \frac{Z_\psi(\alpha_s, \mu)}{\not p}\left[1 - C_F\frac{\alpha_s}{4\pi}\log\frac{\mu^2}{-p^2}\right]\\ &\leftarrow S^R_F(p) = \frac{1}{\not p} \quad \text{for } \mu^2 = -p^2\\ \Gamma^{0,2} &= Z_\psi^{-2/2}\,\Gamma^{0,2}_R(p)\end{aligned}$$

(ii) Vertex:



$$S_F(p')g_{s0}T^a\gamma_\mu S_F(p)D_G^{\mu\nu}(k) \to S'_F(p')g_{s0}T^a\Gamma'_\mu S'_F(p)D'^{\mu\nu}_G(k)$$

$$= Z_\psi^{1/2}S_F^R(p')\left[g_{s0}\frac{Z_\psi Z_G^{1/2}}{Z_1}\right]T^a\Gamma^R_\mu S^R_F(p)Z_\psi^{1/2}D_G^{R\mu\nu}(k)Z_G^{1/2}$$

$$= Z_\psi^{-1/2}S'_F(p')\underbrace{\left[g_{s0}\frac{Z_\psi Z_G^{1/2}}{Z_1}\right]}_{g_s}T^a\Gamma^R_\mu S'_F(p)Z_\psi^{-1/2}D'^{\mu\nu}_G(k)Z_G^{-1/2}$$

$$\Rightarrow g_{s0}\Gamma'_\mu = Z_\psi^{-2/2}Z_G^{-1/2}g_s\Gamma^R_\mu$$

$$\boxed{\Gamma^{N_G N_\psi}(p;g_{s0},\epsilon) = Z_G^{-N_G/2}(g_{s0},\mu)Z_\psi^{-N_\psi/2}(g_{s0},\mu)\Gamma_R^{N_G N_\psi}(p;g_s,\mu)}$$

In gauge theories renormalization constants and g_s are theoretically fixed by 3 Green's functions [mod. gauge/ghosts]:

$$\Gamma_R^{2,0}(p^2=-\mu^2) = Z_G(g_{s0},\mu)\Gamma^{2,0}(p^2=-\mu^2) = -g_{\mu\nu}p^2 + p_\mu p_\nu$$
$$\Gamma_R^{0,2}(p^2=-\mu^2) = Z_\psi(g_{s0},\mu)\Gamma^{0,2}(p^2=-\mu^2) = \not{p}$$
$$\Gamma_R^{1,2}(p^2=-\mu^2) = \sqrt{Z_G}Z_\psi\Gamma^{1,2}(p^2=-\mu^2) = g_s\gamma_\mu$$

RGE: The synchronous variation of μ and $g_s(\mu)$ leaves theory invariant: $\mu \frac{d}{d\mu}\Gamma = 0$

$$\Rightarrow \left\{\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g_s}{\partial\mu}\frac{\partial}{\partial g_s} - \frac{N_G}{2}\mu\frac{\partial \log Z_G}{\partial\mu} - \frac{N_\psi}{2}\mu\frac{\partial \log Z_\psi}{\partial\mu}\right\}\Gamma_R^{N_G,N_\psi}(p;g_s(\mu),\mu) = 0$$

β function: $\beta(g_s) = \mu\frac{\partial}{\partial\mu}g_s(g_{s0},\mu)$

anomalous dimension: $\gamma(g_s) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\log Z(g_{s0},\mu)$

$$\boxed{\left\{\mu\frac{\partial}{\partial\mu} + \beta(g_s)\frac{\partial}{\partial g_s} - N_G\gamma_G(g_s) - N_\psi\gamma_\psi(g_s)\right\}\Gamma_R^{N_G,N_\psi}(p;g_s(\mu),\mu) = 0}$$

Move μ variation to p variation: $\Gamma_R = \mu^D f\left(\frac{p}{\mu}\right)$
[D = physical dimension of Γ_R]

$p \to e^t p$:

$$\boxed{\left\{-\frac{\partial}{\partial t} + \beta(g_s)\frac{\partial}{\partial g_s} + D - N_G\gamma_G(g_s) - N_\psi\gamma_\psi(g_s)\right\}\Gamma_R^{N_G,N_\psi}\left(e^t p;g_s(\mu),\mu\right) = 0}$$

where $t = \log\frac{Q}{\mu}$

Solution: $$\boxed{\begin{aligned}\frac{\partial\bar{g}_s(g_s,t)}{\partial t} &= \beta\left(\bar{g}_s(g_s,t)\right)\\ \bar{g}_s(g_s,0) &= g_s\end{aligned}} \Rightarrow \boxed{t = \int_{g_s}^{\bar{g}_s(g_s,t)}\frac{dg'}{\beta(g')}}$$

- differentiation by t: $1 = \frac{1}{\beta(\bar{g}_s)}\frac{\partial\bar{g}_s}{\partial t}$
- differentiation by g_s:

$$0 = -\frac{1}{\beta(g_s)} + \frac{1}{\beta(\bar{g}_s)}\frac{\partial\bar{g}_s}{\partial g_s} \Rightarrow \beta(g_s)\frac{\partial\bar{g}_s}{\partial g_s} = \beta(\bar{g}_s) = \frac{\partial\bar{g}_s}{\partial t}$$

The most general solution is a function of $\bar{g}_s(g_s, t)$ modified by the special solution determined by the physical and anomalous dimensions:

$$\boxed{\begin{aligned}\Gamma_R^{N_G,N_\psi}\left(e^t p, g_s\right) &= \Gamma_R^{N_G,N_\psi}\left(p, \bar{g}_s(g_s,t)\right)\\ &\quad \exp\left\{Dt - \int_0^t dt' \left[N_G \gamma_G\left(\bar{g}_s(g_s,t')\right) + N_\psi \gamma_\psi\left(\bar{g}_s(g_s,t')\right)\right]\right\}\end{aligned}}$$

$$\gamma_G(g_s) = \left(-\frac{13}{2} + \frac{2}{3}N_F\right)\frac{\alpha_s}{4\pi} + \cdots \qquad \text{[Landau gauge]}$$

$$\gamma_\psi(g_s) = 0 + \cdots$$

$$\frac{\beta(g_s)}{g_s} = -\beta_0 \frac{\alpha_s}{4\pi} - \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \qquad \text{with}$$

$$\left.\begin{aligned}\beta_0 &= 11 - \tfrac{2}{3}N_F\\ \beta_1 &= 102 - \tfrac{38}{3}N_F\end{aligned}\right\}\text{indep. of ren. scheme}$$

(higher orders $\beta_2, \beta_3, \ldots$ depend on ren. scheme)

<u>EFFECTIVE COUPLING</u>

- lowest order:

$$\downarrow g_s^2 = g_s^2(\mu^2)$$

$$t = -\int_{g_s}^{\bar{g}_s} \frac{dg'}{bg'^3} = \frac{1}{2b}\left[\frac{1}{\bar{g}_s^2} - \frac{1}{g_s^2}\right] \Rightarrow \bar{g}_s^2(g_s, t) = \frac{g_s^2}{1 + \left(11 - \frac{2}{3}N_F\right)\frac{g_s^2}{8\pi^2}t}$$

$$t = \frac{1}{2}\log\frac{Q^2}{\mu^2}$$

• higher orders:

$$g_s^2(Q^2) = \frac{(4\pi)^2}{\beta_0 \log \frac{Q^2}{\Lambda^2}} \left[1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log \frac{Q^2}{\Lambda^2}}{\log \frac{Q^2}{\Lambda^2}} + \cdots \right]$$

with $\Lambda^2 = \mu^2 \exp\left\{ -\frac{16\pi^2}{\beta_0\ g_s^2} + \frac{\beta_1}{\beta_0^2} \log(\beta_0\ g_s^2) \right\}$

Q^2 variation of Green's functions:

$$\left.\begin{array}{l} \gamma_G(g_s^2) = -d\, g_s^2 + \cdots \\ d = \frac{1}{16\pi^2}\left(\frac{13}{2} - \frac{2}{3}N_F\right) \\ g_s^2(t) = \frac{g_s^2}{1+2bg_s^2 t} \\ b = \frac{1}{16\pi^2}\left(11 - \frac{2}{3}N_F\right) \end{array}\right\} \quad \begin{aligned} \int_0^t dt' \gamma_G(\bar{g}_s(g_s, t')) &= \int_0^t dt' (-d) \frac{g_s^2}{1+2bg_s^2 t'} \\ &= -\frac{d}{2b} \log(1 + 2bg_s^2 t) \\ &= -\log(1 + 2bg_s^2 t)^{\frac{d}{2b}} \end{aligned}$$

$$\Rightarrow \Gamma_R \propto e^{Dt}\ e^{\log(1+2bg_s^2 t)^{\frac{d}{2b}}} \underset{(t\to\infty)}{\longrightarrow} e^{Dt}\ t^{\frac{d}{2b}}$$

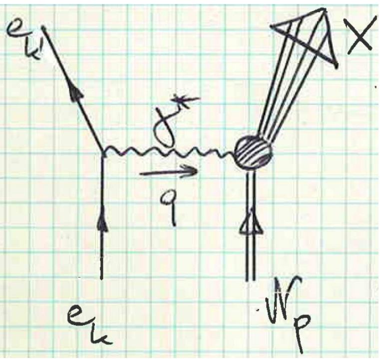
$$\boxed{\Gamma_R \propto Q^D (\log Q)^{\frac{d}{2b}}}$$

Green's functions vary logarithmically with Q^2 in asymptopic free theories.
[← fix point theories: $g = g^* \neq 0 \Rightarrow \Gamma_R \propto Q^D Q^{c^*}$]

B. QCD AT SHORT DISTANCES

§1. Structure Functions of the Nucleon

Asymptotic freedom:
(i) α_s small $\Rightarrow$ 0^{th} approximation: approximately free particles at short distances/high energies
$\Rightarrow$ PARTON MODEL
(ii) $\log Q^2$ dependence through higher orders
[w.l.o.g.: electromagnetic structure functions]



$$\mathcal{M}(X) = ie^2\bar{u}'\gamma^\mu u \frac{1}{q^2}\langle X|j_\mu|\mathcal{N}_p\rangle$$

cross section [$E = e$ lab. energy]

$$d\sigma(e') = \frac{1}{4ME}\frac{d^3k'}{(2\pi)^3 2E'}\frac{1}{4}\sum_X (2\pi)^4\delta_4(p+q-p_X)|\mathcal{M}_X|^2$$

$q = k - k' \quad q^2 = -Q^2 < 0$

$$\frac{1}{4}\sum_X (2\pi)^4\delta_4(p+q-p_X)|\mathcal{M}_X|^2$$

$$= \left(\frac{e^2}{Q^2}\right)^2 \underbrace{\frac{1}{4}\sum_{spins}[\bar{u}'\gamma^\nu u][\bar{u}\gamma^\mu u']}_{=\mathcal{L}^{\mu\nu}\text{ lepton tensor}} \underbrace{\sum_X \langle\mathcal{N}|j_\mu|X\rangle\langle X|j_\nu|\mathcal{N}\rangle(2\pi)^4\delta_4(p+q-p_X)}_{=8\pi W_{\mu\nu}\text{ hadron tensor}}$$

lepton tensor: $\mathcal{L}_{\mu\nu} = k_\mu k'_\nu + k_\nu k'_\mu - (kk')g_{\mu\nu}$ $\leftarrow$ symm. $\mu,\nu; k,k'$

hadron tensor:

$$W_{\mu\nu} = \frac{1}{8\pi}\sum_{spins}\sum_X (2\pi)^4\delta_4(p+q-p_X)\langle\mathcal{N}_p|j_\mu^{elm}|X\rangle\langle X|j_\nu^{elm}|\mathcal{N}_p\rangle$$

$$= \frac{1}{8\pi}\sum_{spins}\int d^4x e^{-iqx}\langle\mathcal{N}_p|\left[j_\mu^{elm}(0), j_\nu^{elm}(x)\right]|\mathcal{N}_p\rangle$$

properties of $W_{\mu\nu}$:

(i) symm. tensor in $p_\mu, q_\mu, g_{\mu\nu}$
(ii) current cons.: $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$ $[\partial^\mu j_\mu^{elm} = 0]$
(iii) tensor real ($\leftarrow$ hermiticity of elm. current)

decomposition in invariants:
general basis: $\underbrace{g_{\mu\nu} \quad q_\mu q_\nu}_{-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}} \qquad \underbrace{p_\mu p_\nu \quad p_\mu q_\nu + p_\nu q_\mu \quad q_\mu q_\nu}_{\left[p_\mu - q_\mu \frac{pq}{q^2}\right]\left[p_\nu - q_\nu \frac{pq}{q^2}\right]}$

$$W_{\mu\nu} = W_1 \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right] + W_2 \left[p_\mu - q_\mu \frac{pq}{q^2}\right]\left[p_\nu - q_\nu \frac{pq}{q^2}\right]$$

W_i = Lorentz scalar structure functions

<u>variables:</u> (i) electron state characterized by energy and scattering angle
(ii) inv.: $Q^2 = -q^2 = 4EE' \sin^2 \frac{\theta}{2}$ scattering angle
$\nu = pq = M(E - E')$ energy loss in e sector

range: $Q^2 \geq 0$, $\nu \geq 0$ $\}$ $(p+q)^2 = W^2 \geq M^2$ (at least $\mathcal{N}$ in final state)
$M^2 + 2pq + q^2 \geq M^2 \Rightarrow 2\nu \geq Q^2$
$= elastic$

(iii) scaling variables:

Bjorken variable	$x = \frac{Q^2}{2\nu}$	$0 \leq x \leq 1$
rel. energy loss	$y = \frac{pq}{pk}$	$0 \leq y \leq 1$

structure fct.: $F_1(x, Q^2) = W_1(\nu, Q^2)$

$F_2(x, Q^2) = \nu W_2(\nu, Q^2)$

<u>Cross section</u> in high-energy limit:

$$\boxed{\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2}{Q^4} s_{e\mathcal{N}} \left[(1-y)F_2(x,Q^2) + y^2 x F_1(x,Q^2)\right]}$$

<u>interpretation of structure functions:</u>

essence of $e\mathcal{N} \to e' +$ evth. is $\underline{\underline{\gamma^* + \mathcal{N} \to \text{evth.}}}$

total absorption cxn of virt. photons

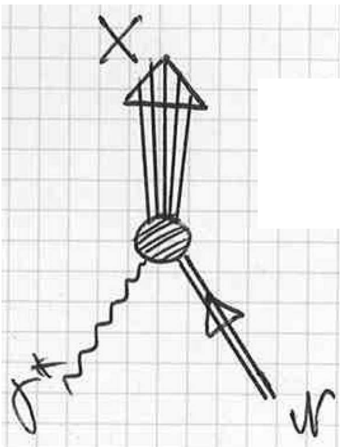
wave function of virt. space-like photons:

$$q_\mu = \left(\frac{\nu}{M}; 0, 0, \sqrt{Q^2 + \frac{\nu^2}{M^2}}\right) \text{ in lab. frame}$$

$\to$ transv. pol.: $\epsilon_\mu(\pm) = \frac{1}{\sqrt{2}}(0; 1, \pm i, 0)$

long. pol.: $\epsilon_\mu(L) = \frac{1}{\sqrt{Q^2}}\left(\sqrt{Q^2 + \frac{\nu^2}{M^2}}; 0, 0, \frac{\nu}{M}\right)$

normalization: $\epsilon_i \epsilon_j^* = \pm\delta_{ij} \quad \epsilon_i q = 0 \quad \epsilon_\pm^* \epsilon_\pm = -1 \quad \epsilon_L^2 = +1$



cxn $\gamma^* + \mathcal{N} \to$ everything

$$\sigma(\gamma^*\mathcal{N}) \propto \sum_X \epsilon^{*\mu} \langle \mathcal{N}|j_\mu|X\rangle\langle X|j_\nu|\mathcal{N}\rangle \epsilon^\nu (2\pi)^4 \delta_4(p+q-p_X)$$

$$\propto \epsilon^{*\mu} W_{\mu\nu} \epsilon^\nu$$

transv. cxn: $\sigma_\pm = \epsilon_\pm^{*\mu} W_{\mu\nu} \epsilon_\pm^\nu = W_1 = F_1 \geq 0$

$[\mathcal{P}_{elm} : \sigma_+ = \sigma_- = \frac{1}{2}\sigma_T]$

long. cxn: $\sigma_L = \epsilon_L^{*\mu} W_{\mu\nu} \epsilon_L^\nu = -W_1 + \left(\frac{\nu^2}{Q^2} + M^2\right) W_2 \geq 0$

$$\underset{(Q^2 \gg M^2)}{\longrightarrow} -F_1 + \frac{1}{2x} F_2$$

<u>R ratio:</u>

$$R = \frac{\sigma_L}{\sigma_T} \qquad R = \left(\frac{\nu^2}{Q^2} + M^2\right)\frac{W_2}{W_1} - 1$$
$$\rightarrow \frac{F_2 - 2xF_1}{2xF_1}$$

<u>Experimental results:</u>

1.) <u>Bjorken scaling:</u>

Bjorken limit: Q^2 large, x fixed $\Big\}$

$$\nu W_2(\nu, Q^2) = F_2(x, Q^2) \rightsquigarrow F_2(x)$$
$$W_1(\nu, Q^2) = F_1(x, Q^2) \underset{Bj}{\rightsquigarrow} F_1(x)$$

scaling most pronounced for $x \sim 0.25$

$$\begin{cases} x \lesssim 0.25 : F_2(x, Q^2) \text{ slightly increasing with } Q^2 \\ x \gtrsim 0.25 : F_2(x, Q^2) \text{ slightly decreasing with } Q^2 \end{cases}$$

small log. violation of scaling predicted by QCD

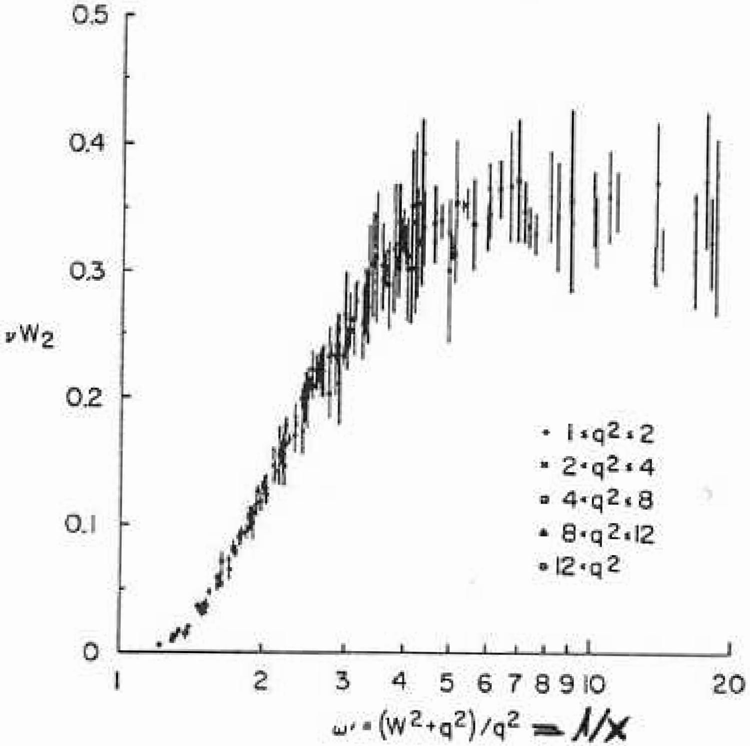
2.) <u>R ratio:</u> $R(x, Q^2) \underset{Bj}{=} \dfrac{F_2(x) - 2xF_1(x)}{2xF_1(x)}$

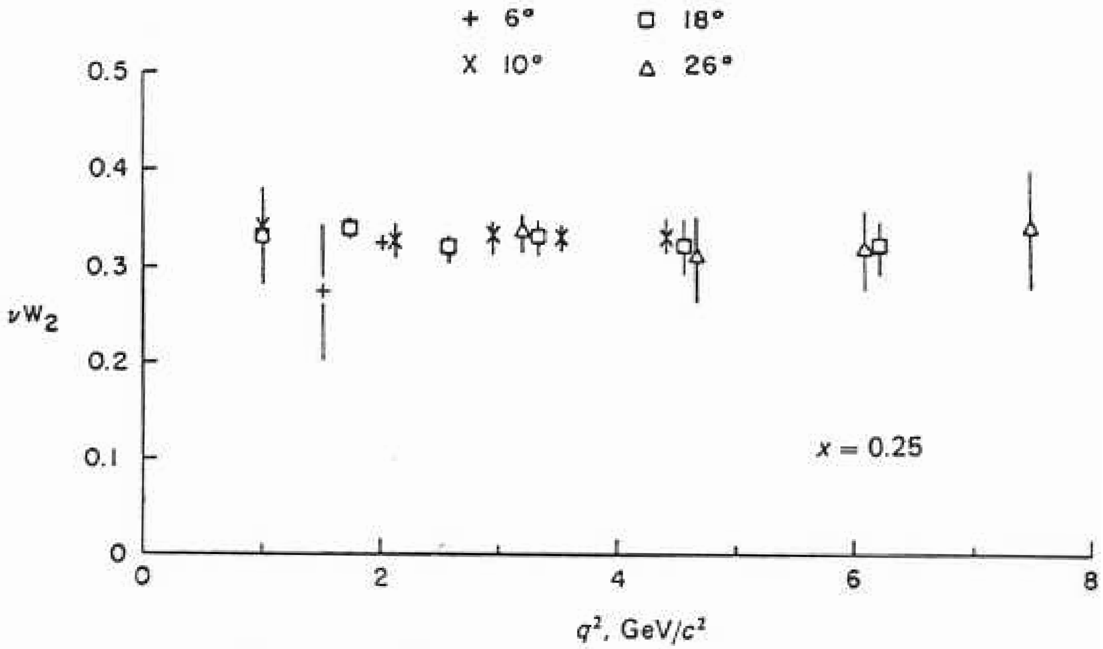
for large Q^2: $R \to 0$, i.e. long. abs. cxn vanishes:

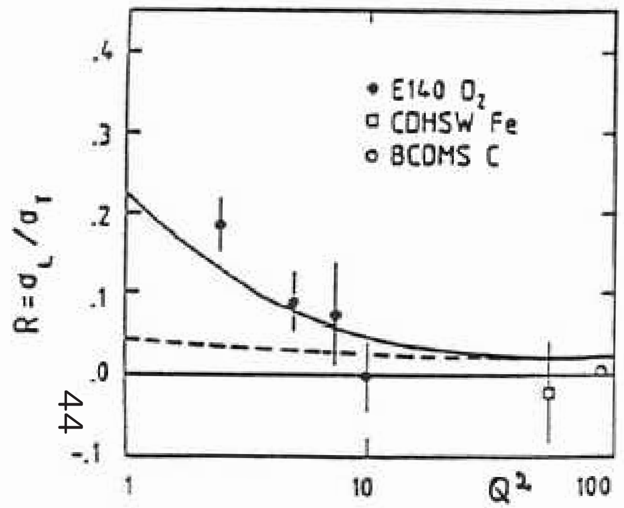
Callan–Gross relation: $F_2 = 2xF_1$

3.) <u>neutron/proton ratio:</u>

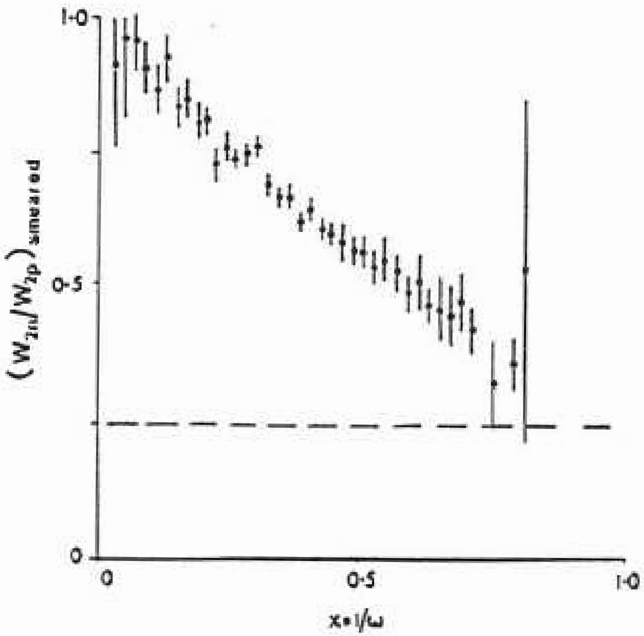
$F_2^N(x)/F_2^P(x)$ decreases from value 1 at $x = 0$ down to a value $\gtrsim \frac{1}{4}$ for $x = 1$.







Neutron / Proton Verhältnis:



Classical quark-parton model

basis:

$e + \text{pt-like} \to e + \text{pt-like}$	$e\mathcal{N} \to e\mathcal{N}$	$e\mathcal{N} \to e + \text{evth.}$
$\frac{d\sigma^{pt}}{dQ^2} \sim \frac{1}{Q^4}$	$\frac{d\sigma^{el}}{dQ^2} \sim \frac{1}{Q^4} \vert F(Q^2)\vert^2 \sim \frac{d\sigma^{pt}}{dQ^2}\left(\frac{M^4}{Q^4}\right)^2$	$\frac{d\sigma}{dQ^2} \sim \frac{1}{Q^4} F_2(x) \sim \frac{d\sigma^{pt}}{dQ^2}$

scaling $F_2(x, Q^2) \approx F_2(x) \Rightarrow$ for $Q^2 \to \infty$ the inclusive cxn behaves analogous to point-like cxn [Q^2 decrease slower by 8 orders than elastic nucleon cxn]

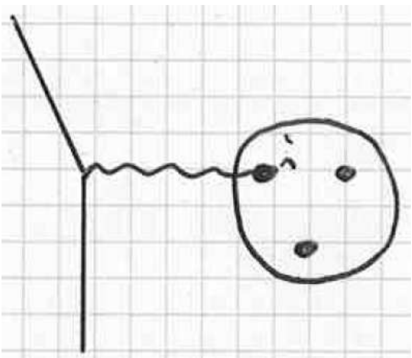
§2. Parton model of deep inelastic lepton-nucleon scattering

In quark picture at high resolution the reaction is built up by superposition of scattering processes off quark constituents: modeling in parton model.

REMARKS

(i) For large Q^2 the superposition is incoherent: $d\sigma = \sum_q d\sigma_q$

classical: $f = \sum_i f_i \quad f_i = \int \frac{d^3\vec{r}}{2\pi} e^{-i\vec{q}\,\vec{r}} V_{\mathcal{C}}(\vec{r} - \vec{r}_i)$



$$= e^{-i\vec{q}\,\vec{r}_i} \int \frac{d^3\vec{r}'}{2\pi} e^{-i\vec{q}\,\vec{r}'} V_{\mathcal{C}}(\vec{r}')$$

$$= e^{-i\vec{q}\,\vec{r}_i} f_{\mathcal{C}}$$

$$d\sigma = d\sigma_R \left| \sum_i e^{-i\vec{q}\,\vec{r}_i} \right|^2 \qquad f = F f_{\mathcal{C}}$$

(a) $|\vec{q}|^{-1} \gg |\vec{r}_i| : d\sigma = N^2 d\sigma_R$

<u>coherent</u> superposition of elem. processes at small Q^2

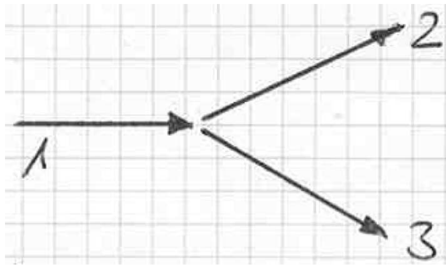
(b) $|\vec{q}|^{-1} \ll |\vec{r}_i| : \sum_{ij} e^{-i\vec{q}\,(\vec{r}_i - \vec{r}_j)} = \sum_{i=j} 1 + \sum_{i \neq j} e^{-i\vec{q}\,(\vec{r}_i - \vec{r}_j)}$

contributions interfer to zero ↑

$$d\sigma = N d\sigma_R$$

<u>incoherent</u> superposition of elem. processes at large Q^2

(ii) probabilistic picture:



"splitting" of a particle 1 into two constituents 2 and 3 for large P

$$p_1 = \left(P + \frac{m_1^2}{2P}; 0_\perp, P \right)$$

$$p_2 = \left(|x|P + \frac{m_2^2 + k_\perp^2}{2|x|P}; k_\perp, xP \right)$$

$$p_3 = \left(|1-x|P + \frac{m_3^2 + k_\perp^2}{2|1-x|P}; -k_\perp, (1-x)P \right)$$

3-momentum conservation

energy jump

Solution of Born series <u>before</u> introduction of time ordering

$$S_{fi} = \lim_{t \to +\infty} \langle f | U(t, -\infty) | i \rangle = \lim_{t \to +\infty} U(t, -\infty)_{fi}$$

$$
\begin{aligned}
U(t,-\infty)_{fi} &= \delta_{fi} + (-i)\int_{-\infty}^{t} dt_1 V_{fi}(t_1) \\
&\quad + (-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \sum_n V_{fn}(t_1) V_{ni}(t_2) + \cdots \\
&= \delta_{fi} + (-i)\int_{-\infty}^{t} dt_1 e^{i(E_f-E_i)t_1} V_{fi} \\
&\quad + (-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \sum_n e^{i(E_f-E_n)t_1 + i(E_n-E_i)t_2} V_{fn} V_{ni} + \cdots \\
&= \delta_{fi} + e^{i(E_f-E_i)t} \left\{ \frac{V_{fi}}{E_i - E_f + i\epsilon} \right. \\
&\quad \left. + \sum_n \frac{V_{fn}}{E_i - E_f + i\epsilon} \, \frac{V_{ni}}{E_i - E_n + i\epsilon} + \cdots \right\}
\end{aligned}
$$

$$
\lim_{t\to\infty} \frac{e^{i(E_f-E_i)t}}{E_i - E_f + i\epsilon} = \frac{1}{i} \lim_{t\to\infty} \int_{-\infty}^{t} dt_1 e^{i(E_f-E_i)t_1} = -2\pi i \delta(E_f - E_i)
$$

$$
S_{fi} = \delta_{fi} - 2\pi i \delta(E_f - E_i) \left\{ V_{fi} + \sum_n \frac{V_{fn} V_{ni}}{E_i - E_n + i\epsilon} + \cdots \right\}
$$

"old-fashioned perturbation theory"

$\Delta E = E_1 - (E_2 + E_3) = P(1 - |x| - |1-x|)$ for $x < 0$ and $x > 1$: $\Delta E \sim P$

$$
= \frac{1}{2P}\left[m_1^2 - \frac{m_2^2 + k_\perp^2}{x} - \frac{m_3^2 + k_\perp^2}{1-x} \right]
$$

for $0 < x < 1$: $\Delta E \sim P^{-1}$

leading

$\Rightarrow$ lifetime $\tau_L \sim \dfrac{1}{\Delta E} \sim \dfrac{P}{\langle k_\perp^2 \rangle}$ very long in eP-c.m.s.

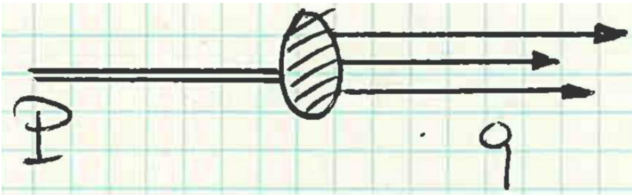
[$x < 0, x > 1$: one of the daughter particles moves backward]

$\Rightarrow$ For fast moving particles the splitting dominates that makes the daughter particles adopt an energy/momentum fraction x with $0 < x < 1$ parallel to the mother particle.

Quark-Parton Model:

(i) In fast moving coordinate systems a nucleon can be split into interaction-free parallel partons that scatter leptons incoherently.

(ii) Partons can be identified with point-like quarks.



Feynman

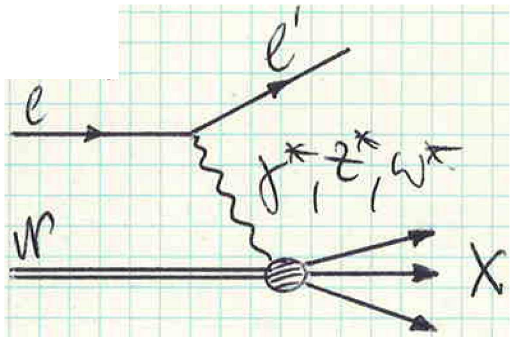
Bjorken, Pachos

Deep Inelastic Lepton-Nucleon Scattering

Lorentz invariance

point-like lepton current

spin 1 exchange $\Rightarrow$



$$\frac{d\sigma^{elm}}{dxdy} = \frac{4\pi\alpha^2}{Q^4} s \left\{(1-y)F_2^{elm}(x,Q^2) + y^2 x F_1^{elm}(x,Q^2)\right\}$$

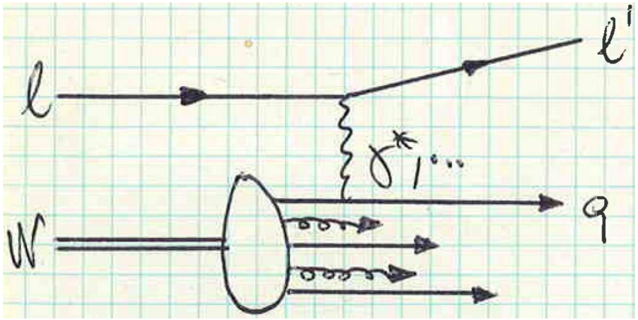
$$\frac{d\sigma_{cc}^{\nu/\bar{\nu}}}{dxdy} = \frac{G_F^2 s}{2\pi} \Big\{(1-y)F_2^{\nu/\bar{\nu}}(x,Q^2) + y^2 x F_1^{\nu/\bar{\nu}}(x,Q^2) \pm \frac{1-(1-y)^2}{2} x F_3^{\nu/\bar{\nu}}(x,Q^2)\Big\}$$

$F_i = F_i(x, Q^2)$ elm. and weak structure functions

transverse: $F_T = F_1$

longitudinal: $F_L = F_2 - 2xF_1 \quad R = \frac{F_L}{2xF_T}$

Quark-Parton Picture:



interaction time:

$$\left.\begin{array}{l} \nu = pq = P(q^0 + q^3) = \dfrac{Q^2}{2x} \\ Q^2 = -(q^0)^2 + (q^3)^2 \end{array}\right\}$$

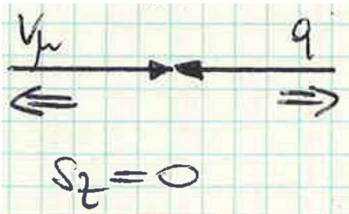
$$\Delta E^* = -q^0 = xP - \frac{Q^2}{4xP} \sim xP \; (x > 0)$$

$$\left.\begin{array}{l} \tau_{int} \sim \dfrac{1}{xP} \\ \tau_L \sim \dfrac{P}{\langle k_\perp^2 \rangle} \end{array}\right\} \boxed{\tau_{int} \ll \tau_L}$$

$\rightarrow$ quarks real in short distance processes

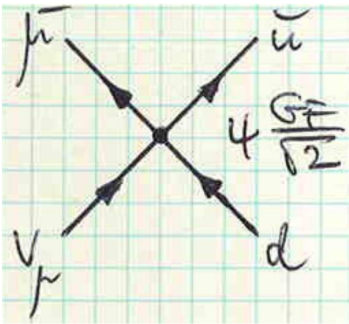
quark-parton cross sections:

$\boxed{\nu_\mu d \rightarrow \mu^- u}$



$$\frac{d\sigma^q}{d\cos\theta_*} \sim \frac{1}{2s_*}\left(\frac{4G_F}{\sqrt{2}}\right)^2 \frac{s_*^2}{8\pi} \sim \frac{G_F^2 s_*}{\pi}$$

$$y = \frac{pq}{pk} \underset{IMF}{=} \frac{p_q q}{p_q k} \underset{CM}{=} \frac{\epsilon_*^2(1-\cos\theta_*)}{2\epsilon_*^2}$$



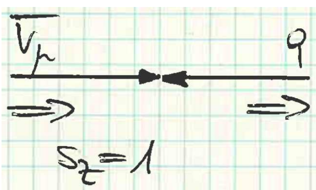
$$y = \frac{1}{2}(1 - \cos\theta_*)$$

$$\frac{d\sigma^q}{dy} = \frac{G_F^2 s_*}{\pi}$$

$\nu_\mu d \rightarrow \mu^- u \quad LL$

$\bar{\nu}_\mu \bar{d} \rightarrow \mu^+ \bar{u} \quad RR$

$\boxed{\bar{\nu}_\mu u \rightarrow \mu^+ d}$



$$\frac{d\sigma^q}{d\cos\theta_*} \sim (1 + \cos\theta_*)^2 \sim (1-y)^2$$

no backw. scatt.: RL, LR $(\nu\bar{q})$

$$\frac{d\sigma^q}{dy} = \frac{G_F^2 s_*}{\pi}(1-y)^2$$

$\boxed{eq \rightarrow eq}$ both helicities incoherent $\quad \dfrac{d\sigma^q}{dy} = \dfrac{2\pi\alpha^2 e_q^2}{Q^4} s_* \left[1 + (1-y)^2\right]$

Composition:

IMF: CM(ℓ,$\mathcal{N}$)
Breit-frame: $q = (0; 0, 0, q)$
etc.

$f_q(\xi)d\xi = \#$ of quarks q in mom. interval $d\xi$ around ξ: $p_q = \xi P$

$$\frac{d\sigma}{dxdy} = \sum_q \int_0^1 d\xi \; f_q(\xi) \; \frac{d\sigma^q(s_* = \xi s)}{dy} \; \delta_1\left(x - \frac{Q^2}{2\nu}\right)$$

$$\frac{Q^2}{2\nu} = \xi \frac{Q^2}{2\nu_q} = \xi$$

$\uparrow$ elasticity condition: $(p_q + q)^2 = p_{q'}^2$

$$-Q^2 + 2qp_q = 0 \Rightarrow \frac{Q^2}{2\nu_q} = 1$$

$$\boxed{\frac{d\sigma}{dxdy} = \sum_q f_q(x) \frac{d\sigma^q(s_* = xs)}{dy}}$$

Bjorken-variable determines the relative momentum of the scattered quark: $\xi = x$

$$\nu: \quad \frac{d\sigma}{dxdy} = \frac{G_F^2 s}{\pi}\left\{xd(x) + (1-y)^2 x\bar{u}(x)\right\}$$

$$\bar{\nu}: \quad \frac{d\sigma}{dxdy} = \frac{G_F^2 s}{\pi}\left\{(1-y)^2 xu(x) + x\bar{d}(x)\right\}$$

$$e: \quad \frac{d\sigma}{dxdy} = \frac{2\pi\alpha^2 s}{Q^4} \sum_q e_q^2 x f_q(x) \left[1 + (1-y)^2\right]$$

Analysis:

$F_i(x, Q^2)$ independent of Q^2: scaling

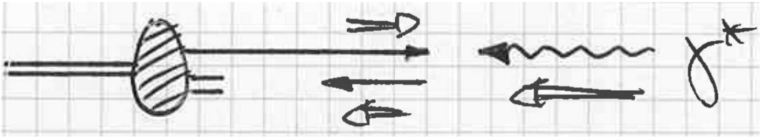
$F_2 = 2xF_1$ Callan–Gross relation

$$F_2^{elm} = \sum_q e_q^2 x f_q(x) \qquad F_2^{\nu} = 2x(d + \bar{u}) \qquad xF_3^{\nu} = +2x(d - \bar{u})$$

$$F_2^{\bar{\nu}} = 2x(u + \bar{d}) \qquad xF_3^{\bar{\nu}} = -2x(u - \bar{d})$$

Physical interpretation:

1.) Callan–Gross relation measures quark spin $= \frac{1}{2}$



Breit frame $q = (0; 0, 0, q)$: current $\bar{q}\gamma_\mu q$ chirally conserved

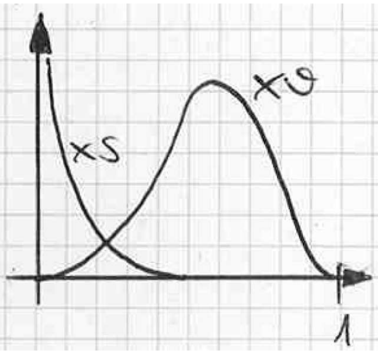
massless parallel quarks

$\Delta S_z = 1 \Rightarrow S_z(\gamma^*) = 1, \neq 0$

$\Rightarrow \sigma_T \neq 0, \sigma_L = 0 \Rightarrow R = \dfrac{\sigma_L}{\sigma_T} = 0$

[spinless partons: $\sigma_T = 0, \sigma_L \neq 0 \Rightarrow R = \infty$ ↯]

2.) valence quarks: $f = v + s$ $\quad v(x)$ valence distribution



$$\int_0^1 dx\ v_d = 1$$

$$\int_0^1 dx\ v_u = 2$$

$v(x) \sim x^{-1/2}(1-x)^3$

$s(x)$ div. sea quarks from quantum fluct.

$$\int_\epsilon^1 dx\ s(x) \sim \log \epsilon$$

$s(x) \sim x^{-1}(1-x)^{7\ldots10} \approx 0$ for $x \gtrsim 0.2$

fractionized electric quark charge:

nuclear target
valence region

$\mathcal{N} = \frac{1}{2}(P + N)$

$$F_2^{elm} \approx x\left[\frac{4}{9}\frac{u+d}{2} + \frac{1}{9}\frac{d+u}{2}\right] = \frac{5}{18}x(u+d)$$

$$F_2^{\nu} \approx 2x\frac{d+u}{2} = x(u+d)$$

$$\boxed{F_2^{elm} \approx \tfrac{5}{18}F_2^{\nu}}$$

3.) 3 quarks in nucleon

nuclear target

val. dominance

$$\sigma_\nu \approx \frac{G_F^2 s}{\pi} \int_0^1 dx\; x\frac{u+d}{2} \qquad \sigma_{\bar\nu} \approx \frac{1}{3}\sigma_\nu$$

$$\sigma_{\bar\nu} \approx \frac{1}{3}\,\frac{G_F^2 s}{\pi} \int_0^1 dx\; x\frac{u+d}{2}$$

↑ spin: quarks no antiquarks

sum rules: ("exact")

baryon number $1 = \int_0^1 dx\; \frac{1}{3}\left[(u-\bar u) + (d-\bar d) + (s-\bar s)\right]$

isospin $\pm\frac{1}{2} = \int_0^1 dx \left[\frac{1}{2}(u-\bar u) - \frac{1}{2}(d-\bar d)\right]$

strangeness $0 = \int_0^1 dx\; (s-\bar s)$

solution proton: $\int_0^1 dx\;(u-\bar u) = 2 \quad \int_0^1 dx\;(d-\bar d) = 1 \quad \int_0^1 dx\;(s-\bar s) = 0$

nuclear target: $\int_0^1 dx\; F_3^\nu = \int_0^1 dx\;[d+u-\bar u-\bar d] = \int_0^1 dx\left[(u-\bar u)+(d-\bar d)\right]$

Gross–Llewellyn-Smith: $\int_0^1 dx\;F_3^\nu = 3$

4.) momentum sum rule: $1 = \sum_{q,\bar q}\int_0^1 d\xi\;\xi f_q(\xi) + \int_0^1 d\xi\;\xi f_g(\xi)$

flavor-neutral matter: ↑

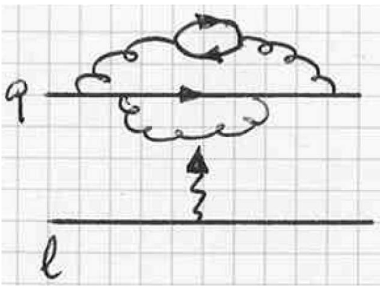
binding energy

measurement: $\int_0^1 d\xi\;\xi f_g(\xi) \approx \frac{1}{2}$

50% of the nucleon energy in fast moving particles is carried by flavor-neutral binding energy: GLUONS

§3. Scaling Violation: Altarelli–Parisi Equations (DGLAP)

idea: parton-quarks are surrounded by a gluon cloud inside nucleon;

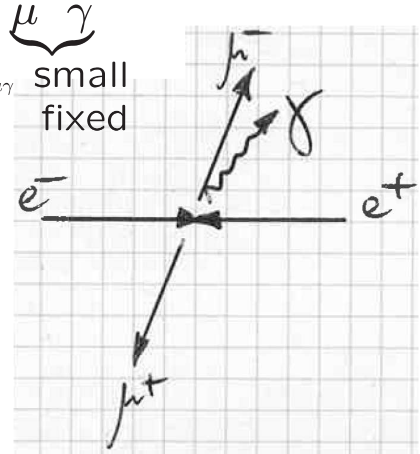


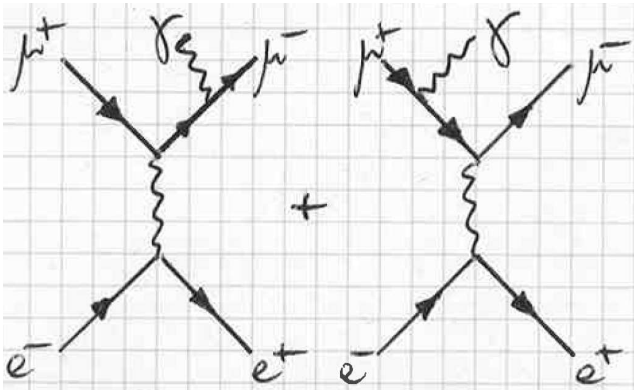
at sufficiently large Q^2 more and more quantum fluctuations are resolved $\Rightarrow$ momentum spectra of quarks and gluons vary with Q^{-1}: microscopic parton distributions are Q^2-dependent.

splitting probability:

$e^+e^- \to \mu^+ \underbrace{\mu^-\gamma}$

$\theta_{\mu\gamma}$ small fixed





$$x_{1,2} = \frac{E_\pm}{E} \qquad z = \frac{E_\gamma}{E}$$

$$x_\perp = \frac{2}{x_1}\sqrt{(1-x_1)(1-x_2)(1-z)} = \frac{p_\perp}{E}$$

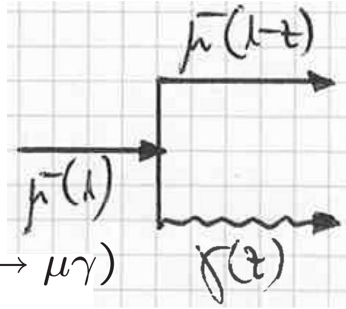
$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dx_1 dx_2} = \frac{\alpha}{2\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

$$\log x_\perp^2 \approx \log(1-x_1)$$

$$d\log p_\perp^2 \approx \frac{dx_1}{1-x_1}$$

$$x_1 + x_2 + z = 2$$

Fragmentation: $x_2 \approx 1 - z$



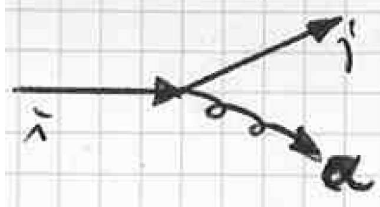
$$d\sigma = \sigma_0 \int^{Q^2} \frac{dp_\perp^2}{p_\perp^2}\frac{\alpha}{2\pi}\frac{1+(1-z)^2}{z}dz$$

cxn = μ-pair cxn $*$ particle flux ($\mu \to \mu\gamma$)

increase of particle flux at $Q^2 \to Q^2 + \delta Q^2$:

$$\frac{\delta N(\mu \to \mu\gamma)}{\delta \log Q^2} = \frac{\alpha}{2\pi}\frac{1+(1-z)^2}{z}dz$$

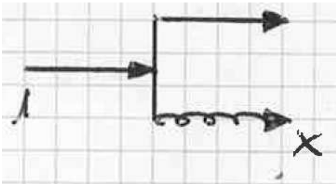
quark fragmentation: color average/sum



$$\sum_{k,a} T^a_{ik} T^a_{kj} = \frac{4}{3}\delta_{ij}$$

QCD splitting probabilities: $\dfrac{\delta N}{\delta \log \frac{Q^2}{\Lambda^2}} = \dfrac{\alpha_s(Q^2)}{2\pi} P(x)dx$

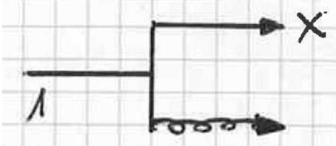
$q \to q + g(x)$



$$P_{gq} = \frac{4}{3}\,\frac{1+(1-x)^2}{x}$$

bremsstrahl-sing. $x \to 0$

$q \to q(x) + g$



$$P_{qq} = \frac{4}{3}\,\frac{1+x^2}{1-x}$$

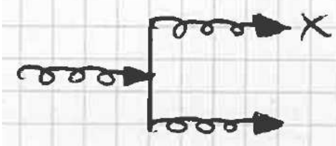
bremsstrahl-sing. $x \to 1$

$g \to q(x) + \bar{q}$

$$P_{qg} = P_{\bar{q}g} = \frac{1}{2}[x^2 + (1-x)^2]$$

finite

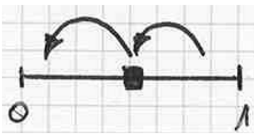
$g \to g(x) + g$



$$P_{gg} = 6\,\frac{[1-x+x^2]^2}{x(1-x)}$$

bremsstrahl-sing. $x \to 0, 1$

Altarelli–Parisi master equations for parton densities:



$Q^2 \to Q^2 + \delta Q^2$

$$\frac{\partial q(x,Q^2)}{\partial \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\ \delta_1(x-yz)\left\{P_{qq}(y)q(z,Q^2) + P_{qg}(y)g(z,Q^2)\right\} - \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy'\ P_{qq}(y')q(x,Q^2)$$

$$\int_0^1 dy' P_{qq}(y') q(x,Q^2) = \int_0^1 dy \int_0^1 dz \delta_1(x-yz)\delta(y-1)\left[\int_0^1 dy' P_{qq}(y')\right] q(z,Q^2)$$

$$
\begin{aligned}
\frac{\partial q(x,Q^2)}{\partial \log Q^2} &= \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\ \delta_1(x-yz)\left\{P_{qq}^R(y)q(z,Q^2)\right. \\
&\qquad\qquad \left.+P_{qg}(y)g(z,Q^2)\right\} \\
\frac{\partial g(x,Q^2)}{\partial \log Q^2} &= \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\ \delta_1(x-yz)\left\{P_{gq}(y)\sum_{fl}[q(z,Q^2)+\bar{q}(z,Q^2)]\right. \\
&\qquad\qquad \left.+P_{gg}^R(y)g(z,Q^2)\right\} \\
P_{qq}^R(y) &= P_{qq}(y)-\delta(y-1)\int_0^1 dy' P_{qq}(y') \\
P_{gg}^R(y) &= P_{gg}(y)-\delta(y-1)\left[\frac{1}{2}\int_0^1 dy' P_{gg}(y') + N_F\int_0^1 dy' P_{qg}(y')\right] \\
\alpha_s(Q^2) &= \frac{12\pi}{(33-2N_F)\log\frac{Q^2}{\Lambda^2}}
\end{aligned}
$$

partial disentanglement: $\delta = q - q'$ non-singlet

$$\left.\begin{array}{l}\Sigma = \sum_{fl}(q+\bar{q}) \\ g\end{array}\right\}\ \text{coupled singlet set}$$

SOLUTIONS:

transition to moments: $q(N,Q^2) = \int_0^1 dx\ x^{N-1} q(x,Q^2)$

transforms integro-differential system of equations into system of usual differential equations.

natural variable: $s = \log\frac{\log Q^2}{\log Q_0^2}$

[$Q_0 =$ reference momentum transfer]

[for fixed coupling constant $t = \log Q^2$ would be the natural variable]

1.) Non-singlet density:

$$\frac{\partial}{\partial s}\delta(N,Q^2) = \frac{6}{33-2N_F}\int_0^1 dy\ y^{N-1}P_{qq}^R(y)\delta(N,Q^2)$$

$$\uparrow = \frac{6}{33-2N_F}\frac{4}{3}\left[-\frac{1}{2}+\frac{1}{N(N+1)}-2\sum_{j=2}^{N}\frac{1}{j}\right] \equiv -d_{NS}(N)$$

$$\frac{\partial}{\partial s}\delta(N,Q^2) = -d_{NS}(N)\delta(N,Q^2) \Rightarrow \delta = \delta_0 e^{-sd_{NS}}$$

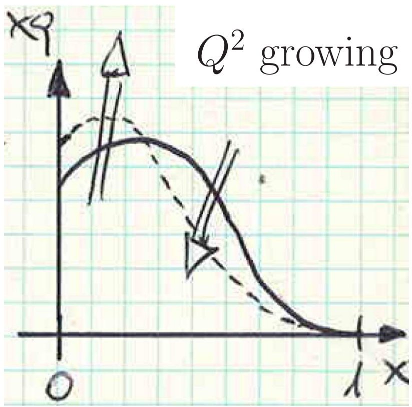
$$\boxed{\begin{aligned}\delta(N,Q^2) &= \delta(N,Q_0^2)\left[\frac{\log Q^2}{\log Q_0^2}\right]^{-d_{NS}}\\ &= \delta(N,Q_0^2)\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{d_{NS}}\end{aligned}}$$

$\leftarrow$ log. violation of Bjorken scaling

interpretation:

(i) asymptotic freedom $\Rightarrow \left[\frac{\log Q^2}{\log Q_0^2}\right]^{-d}$

fixed coupling $\Rightarrow \left[\frac{Q^2}{Q_0^2}\right]^{-d}$



(ii) $d_{NS}(N=1)=0$: net quark # unchanged

$d_{NS}(N>1)>0$: moments decrease with increasing Q^2

(iii) moment comparison: test of anomalous dimensions

Q^2-dependence of structure functions

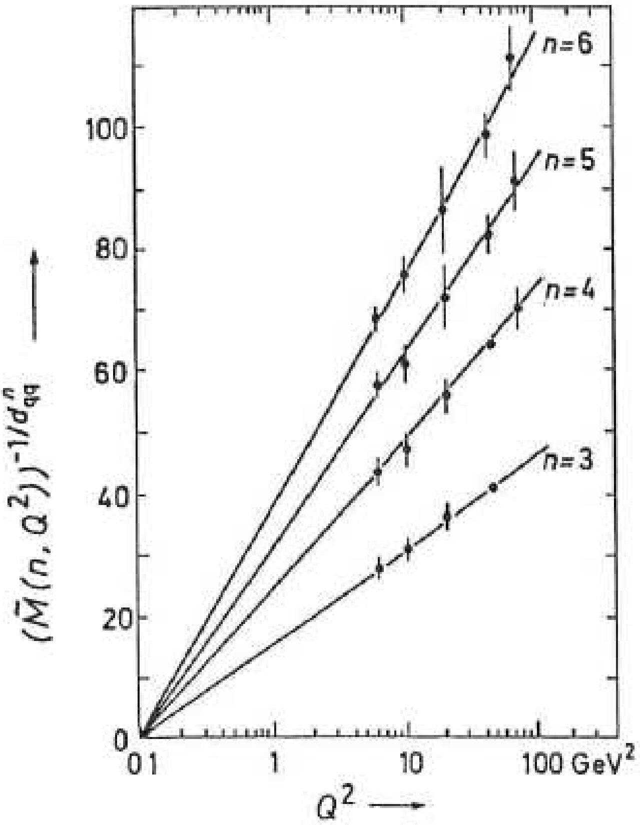


Bild 19-7
Die Momente der Strukturfunktion $F_3^{(\nu N)}$, gemessen in Neutrino-Eisen-Streuung (nach de Groot 1979)

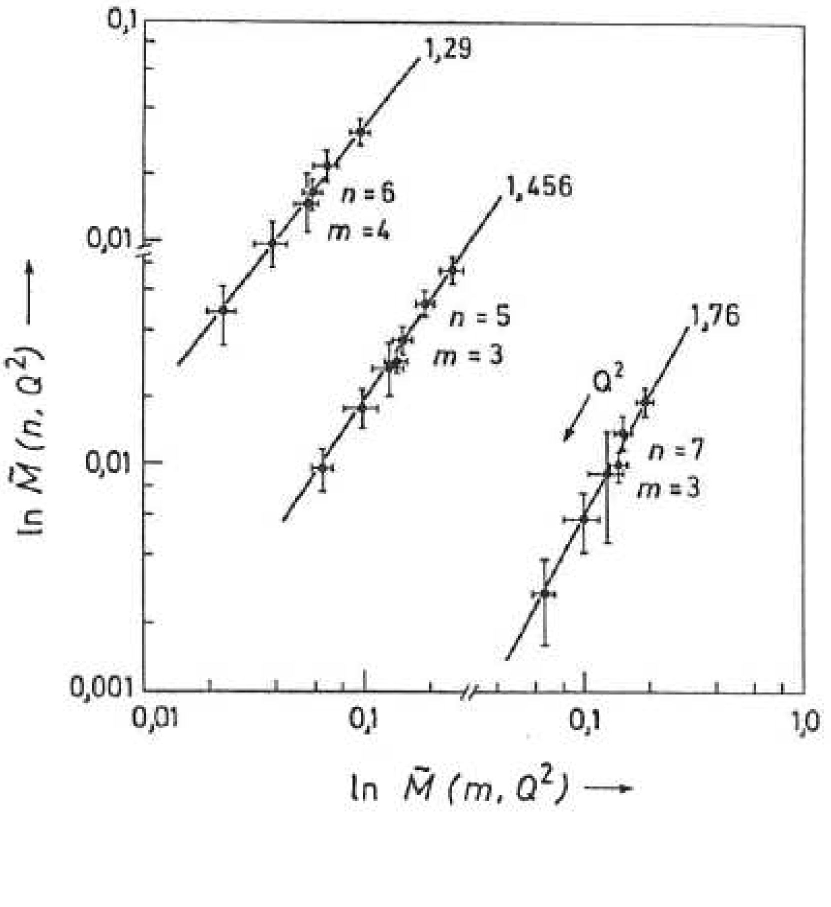


Bild 19-8
Logarithmen von Momenten der Strukturfunktion F_3 gegeneinander aufgetragen. Die QCD-Vorhersagen sind gerade Linien mit berechenbarem Anstieg, wie angegeben (nach Bosetti 1978).

2.) Quark singlet and gluon densities:

$$\frac{\partial}{\partial s}\begin{pmatrix}\Sigma\\ G\end{pmatrix} = -\begin{pmatrix} d_{QQ} & d_{QG}\\ d_{GQ} & d_{GG}\end{pmatrix}\begin{pmatrix}\Sigma\\ G\end{pmatrix} \quad \text{with } \Sigma = \Sigma(N,Q^2) \text{ etc.}$$

$$d_{QQ}(N) = -\frac{6}{33-2N_F}\int_0^1 dy\; y^{N-1} P^R_{qq}(y)$$
$$= \frac{4}{33-2N_F}\left[1 - \frac{2}{N(N+1)} + 4\sum_{j=2}^{N}\frac{1}{j}\right] \equiv d_{NS}(N)$$

$$d_{QG}(N) = -\frac{6}{33-2N_F}\int_0^1 dy\; y^{N-1} 2N_F\, P_{qg}(y) = -\frac{6N_F}{33-2N_F}\frac{N^2+N+2}{N(N+1)(N+2)}$$

$$d_{GQ}(N) = -\frac{6}{33-2N_F}\int_0^1 dy\; y^{N-1} P_{gq}(y) = -\frac{8}{33-2N_F}\frac{N^2+N+2}{(N-1)N(N+1)}$$

$$d_{GG}(N) = -\frac{6}{33-2N_F}\int_0^1 dy\; y^{N-1} P^R_{gg}(y)$$
$$= \frac{9}{33-2N_F}\left\{\frac{1}{3} - \frac{4}{N(N-1)} - \frac{4}{(N+1)(N+2)} + 4\sum_{j=2}^{N}\frac{1}{j} + \frac{2N_F}{9}\right\}$$

solution of the systems via exponential ansatz $\Rightarrow$

$$\boxed{\begin{aligned}\Sigma &= \frac{1}{\mu_+ - \mu_-}\left\{[-\mu_-\Sigma_0 + G_0]e^{-d_+ s} + [\mu_+\Sigma_0 - G_0]e^{-d_- s}\right\}\\ G &= \frac{1}{\mu_+ - \mu_-}\left\{\mu_+[-\mu_-\Sigma_0 + G_0]e^{-d_+ s} + \mu_-[\mu_+\Sigma_0 - G_0]e^{-d_- s}\right\}\end{aligned}}$$

eigenvalues: $d_\pm(N) = \frac{1}{2}\left[(d_{GG}+d_{QQ}) \pm \sqrt{(d_{GG}-d_{QQ})^2 + 4d_{QG}d_{GQ}}\right]$

eigenvectors: $\mu_\pm(N) = \frac{d_\pm - d_{QQ}}{d_{QG}}$

$$= \frac{1}{2}\frac{d_{GG} - d_{QQ} \pm \sqrt{(d_{GG}-d_{QQ})^2 + 4d_{QG}d_{GQ}}}{d_{QG}}$$

PHYSICAL CONCLUSIONS:

(a) momentum sum rule:

$$\left.\begin{array}{l} d_-(2)=0 \\ \mu_+(2)=-1 \end{array}\right\} \underline{\underline{\Sigma(2)+G(2)=1}} \text{ follows from } \Sigma_0(2)+G_0(2)=1$$

(b) asymptotic momentum distribution:

$$\left.\begin{array}{l} d_-(2)=0 \\ \mu_+(2)=-1 \end{array}\right\} \begin{array}{l} \Sigma(2) \to \dfrac{1}{\mu_-(2)-\mu_+(2)} = \dfrac{3N_F}{16+3N_F} = \dfrac{3}{7} \text{ for } N_F=4 \\ \underline{\underline{G(2) \to \dfrac{\mu_-(2)}{\mu_-(2)-\mu_+(2)} = \dfrac{16}{16+3N_F} = \dfrac{4}{7}}} \text{ for } N_F=4 \end{array}$$

(c) measurement of all gluon moments:

deep inelastic $\ell\mathcal{N}$ scatt.: no dir. poss. to meas. gluon density.

indirect:
- momentum sum rule: $G(2,Q_0^2)=1-\Sigma(2,Q_0^2)$
- modifies strength of $\Sigma(N,Q^2)$-variation with Q^2 through buildup of sea density.
- [$G \to Q\bar{Q}$ splitting into heavy quarks]

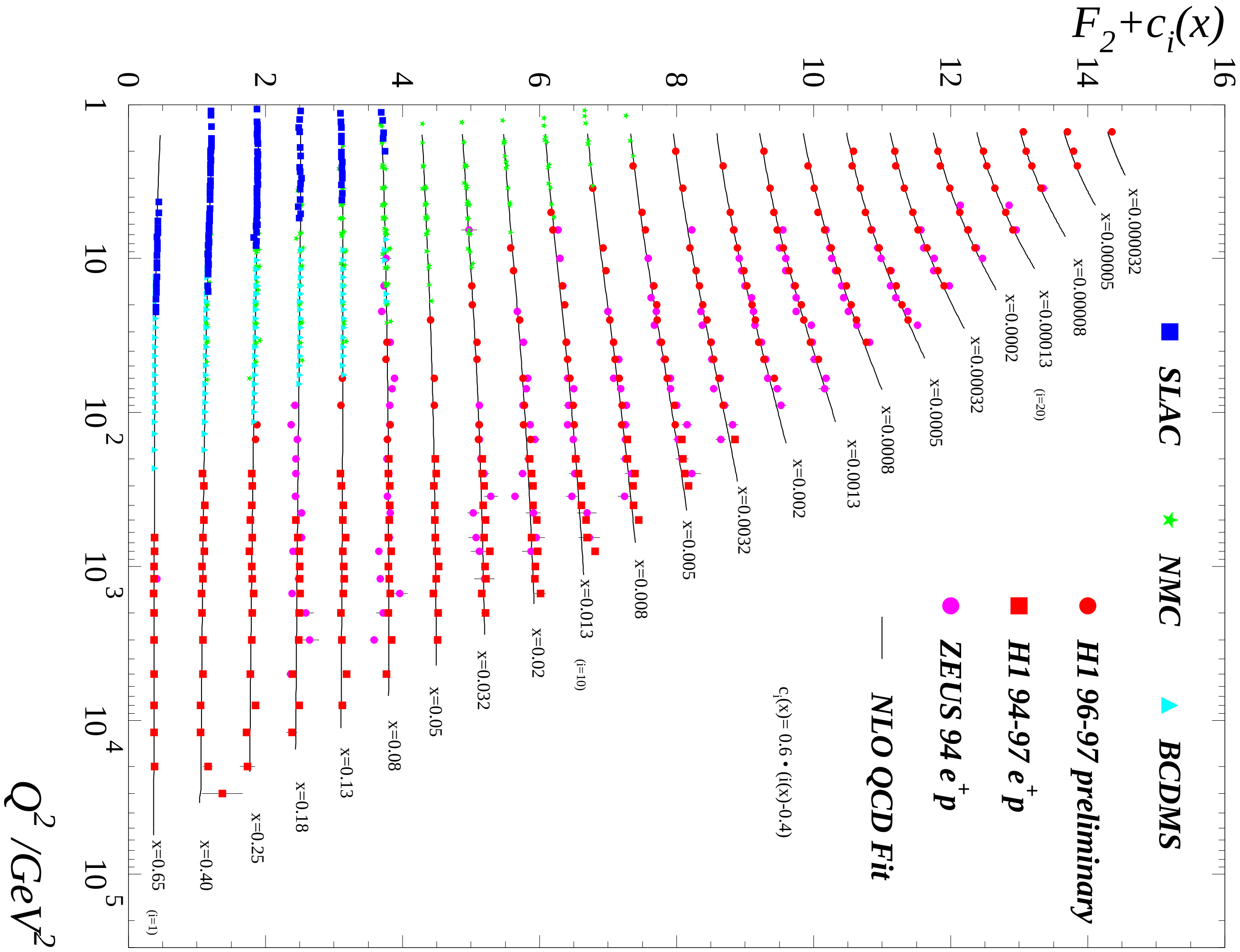
$$F_2^S(x,Q^2) = \frac{5}{18}x[u+\bar{u}+d+\bar{d}] \text{ w.l.o.g. } = \frac{5}{18}x\Sigma(x,Q^2)$$

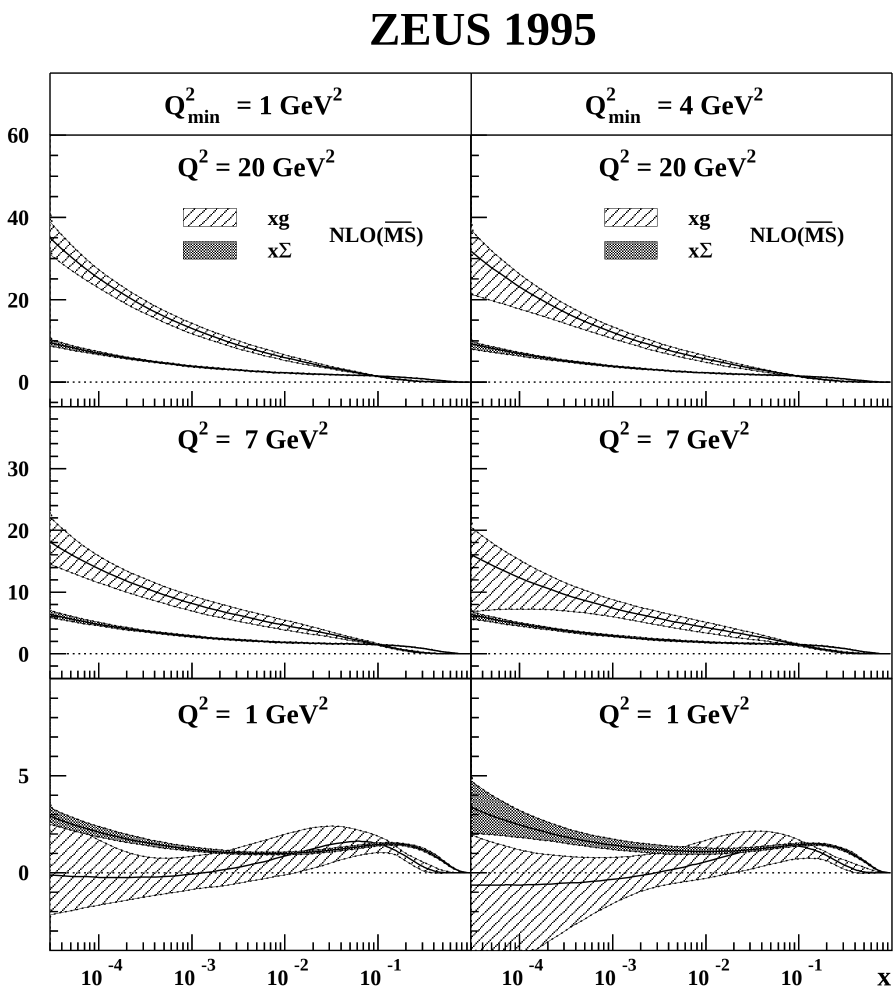
$$F_2^S(N-1,Q^2) = \frac{5}{18}\Sigma(N,Q^2)$$

$$= \frac{5}{18}\left\{ \frac{-\mu_- e^{-d_+ s}+\mu_+ e^{-d_- s}}{\mu_+-\mu_-} \underbrace{\Sigma(N,Q_0^2)}_{=\frac{18}{5}F_2^S(N-1,Q_0^2)} + \frac{e^{-d_+ s}-e^{-d_- s}}{\mu_+-\mu_-}G(N,Q_0^2) \right\}$$

$$\boxed{\frac{1}{A_N(s)}F_2^S(N-1,Q^2) = F_2^S(N-1,Q_0^2) + \frac{B_N(s)}{A_N(s)}G(N,Q_0^2)}$$

Left side determined as straight line in $B_N(s)/A_N(s)$ with G density as slope


$F_2+c_i(x)$
Q^2 /GeV^2
SLAC
NMC
BCDMS
H1 96-97 preliminary
H1 94-97 e+p
ZEUS 94 e+p
NLO QCD Fit
$c_i(x) = 0.6 \cdot (i(x)-0.4)$
x=0.000032
x=0.00005
x=0.00008
x=0.00013 (i=20)
x=0.0002
x=0.00032
x=0.0005
x=0.0008
x=0.0013
x=0.002
x=0.0032
x=0.005
x=0.008
x=0.013 (i=10)
x=0.02
x=0.032
x=0.05
x=0.08
x=0.13
x=0.18
x=0.25
x=0.40
x=0.65 (i=1)


ZEUS 1995
Q^2_{min} = 1 GeV^2
Q^2_{min} = 4 GeV^2
Q^2 = 20 GeV^2
Q^2 = 7 GeV^2
Q^2 = 1 GeV^2
xg
xΣ
NLO($\overline{MS}$)
x

§4. Factorization Theorems of QCD

QCD corrections to deep inelastic $\ell\mathcal{N}$ scattering:

DIMENSIONAL REGULARIZATION

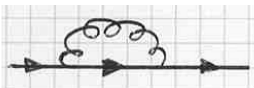
idea: analytical continuation 4-dim. $\to$ n-dim. $[n = 4 - 2\epsilon]$

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^nk}{(2\pi)^n}$$

divergent integral: $\displaystyle\int \frac{d^4k}{k^4} \to \int \frac{d^nk}{k^4} \propto \frac{1}{n-4} = -\frac{1}{2\epsilon}$

$\Rightarrow$ UV singularities as poles for $\epsilon \to 0+$
[gauge invariance preserved]

Feynman parametrization

 $\displaystyle\sim \int \frac{d^nq}{q^2[(q-p)^2 - m^2]}$ simpler treatment of integrals with one denominator

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1-x)]^2} = -\frac{1}{A-B}\left[\frac{1}{A} - \frac{1}{B}\right] = \frac{1}{AB}$$

$$= \int_0^1 dx \int_0^1 dy \frac{\delta_1(x+y-1)}{[Ax+By]^2}$$

$$\boxed{\frac{1}{\prod_{i=1}^{N} A_i} = \Gamma(N) \int_0^1 dx_1 \cdots dx_N \frac{\delta_1(\sum x_i - 1)}{[\sum A_i x_i]^N}}$$

more formulae through differentiation by A_i

Basic integrals of dimensional regularization:

$$(*)\int \frac{d^nk}{(2\pi)^n}\frac{1}{[k^2+2kQ-M^2]^\alpha} = \frac{i(-1)^\alpha}{\Gamma(\alpha)(4\pi)^{n/2}}\;\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{[Q^2+M^2]^{\alpha-\frac{n}{2}}}$$

$$M^2 \equiv M^2 - i\bar{\epsilon}$$

$$\Gamma(x) \approx \frac{1}{x} \text{ for } x \to 0$$

hence derivable by differentiation:

$$\int \frac{d^nk}{(2\pi)^n}\frac{k^\mu}{[k^2+2kQ-M^2]^\alpha} = \frac{i(-1)^{\alpha+1}}{\Gamma(\alpha)(4\pi)^{n/2}}\;\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{[Q^2+M^2]^{\alpha-\frac{n}{2}}}(-Q^\mu)$$

$$\int \frac{d^nk}{(2\pi)^n}\frac{k^\mu k^\nu}{[k^2+2kQ-M^2]^\alpha} = \frac{i(-1)^{\alpha}}{\Gamma(\alpha)(4\pi)^{n/2}}\left\{\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{[Q^2+M^2]^{\alpha-\frac{n}{2}}}Q^\mu Q^\nu - \frac{\Gamma\left(\alpha-1-\frac{n}{2}\right)}{[Q^2+M^2]^{\alpha-1-\frac{n}{2}}}\frac{g^{\mu\nu}}{2}\right\}$$

surface of n-dim. sphere: $\Omega_n = \dfrac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$ $\qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\qquad \Gamma(N) = (N-1)!$

[from $\left\{\int_{-\infty}^{\infty} dx\; e^{-x^2}\right\}^n$ calculated in cartesian and spherical coordinates]

Proof (∗):

$$\int d^nk = \int_{-\infty}^{\infty} dk_0 \int_0^\infty d\omega\;\omega^{n-2}\int d\Omega_{n-1}$$

$$I_n(Q) = \int \frac{d^nk}{(2\pi)^n}\frac{1}{[k^2+2kQ-M^2]^\alpha} \underset{(k\to k-Q)}{=} \int \frac{d^nk}{(2\pi)^n}\frac{1}{[k^2-(Q^2+M^2)]^\alpha}$$

$$= \frac{2\pi^{-1/2}}{(4\pi)^{n/2}\Gamma\left(\frac{n-1}{2}\right)}\int_{-\infty}^{\infty} dk_0 \int_0^\infty \frac{d\omega\;\omega^{n-2}}{[k_0^2-\omega^2-(Q^2+M^2)]^\alpha}$$

Euler function: $B(x,y) = 2\displaystyle\int_0^\infty dt\; t^{2x-1}(1+t^2)^{-x-y} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

$$I_n(Q) = \frac{2\Gamma\left(\alpha-\frac{n-1}{2}\right)}{(4\pi)^{n/2}\sqrt{\pi}\,\Gamma(\alpha)}\int_0^\infty dk_0 \frac{(-1)^\alpha}{[Q^2+M^2-k_0^2]^{\alpha-\frac{n-1}{2}}}$$

$$= \frac{i(-1)^\alpha}{\Gamma(\alpha)(4\pi)^{n/2}}\;\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{[Q^2+M^2]^{\alpha-\frac{n}{2}}} \qquad \text{q.e.d.}$$

Clifford algebra in n dimensions:

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$ [$\mathbb{1}$ = 4-dim. unit-matrix]

possible in n dimensions

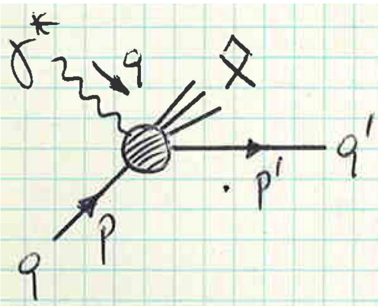
$\Rightarrow Tr\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$

$\gamma^\mu\gamma_\mu = g^\mu_\mu = n$

$\gamma^\mu \not{a}\gamma_\mu = 2a^\mu\gamma_\mu - \not{a}\gamma^\mu\gamma_\mu = (2-n)\not{a}$ etc.

up to anomalies γ_5 can be treated as usual $\left.\right\}$ $\{\gamma^\mu, \gamma_5\} = 0$, $\gamma_5^2 = \mathbb{1}$

deep inelastic $\ell\mathcal{N}$ scattering:



$$z = \frac{-q^2}{2pq} = \frac{Q^2}{2pq} \Rightarrow -\frac{pq}{q^2} = \frac{1}{2z}$$

$$q = p' - p + p_{\hat{X}}$$

parton tensor:

$$\hat{W}^{\mu\nu} = \hat{F}_1(z,Q^2)\left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right] + \frac{2z}{Q^2}\hat{F}_2(z,Q^2)\left(p^\mu + \frac{q^\mu}{2z}\right)\left(p^\nu + \frac{q^\nu}{2z}\right)$$

$$\Rightarrow p^\mu p^\nu \hat{W}_{\mu\nu} = \frac{Q^2}{4z^2}\left(\frac{\hat{F}_2}{2z} - \hat{F}_1\right) \equiv \frac{Q^2}{4z^2}\frac{\hat{F}_L}{2z}$$

$$-g^{\mu\nu}\hat{W}_{\mu\nu} = (1-\epsilon)\frac{\hat{F}_2}{z} - \frac{3-2\epsilon}{2z}\hat{F}_L$$

$$\boxed{\hat{\mathcal{F}}_1(z,Q^2) = \hat{F}_1(z,Q^2) \qquad \hat{\mathcal{F}}_{2,L}(z,Q^2) = \frac{\hat{F}_{2,L}(z,Q^2)}{2z}}$$

structure functions: $\mathcal{F}_i(x,Q^2) = \sum_{q,\bar{q}} e_q^2 \int_0^1 dydz\ \hat{\mathcal{F}}_i(z,Q^2)\ q(y,Q^2)\ \delta(x-yz)$

1.) <u>Born term:</u> $\mathcal{M}^\mu_{LO} = -iee_q\delta_{ij}\bar{u}(p')\gamma^\mu u(p)$

$$\hat{W}^{\mu\nu}_{LO} = \frac{1}{N_c}\sum \mathcal{M}^\mu_{LO}\mathcal{M}^{*\nu}_{LO}\frac{dPS_1(p+q;p')}{8\pi\sigma_0}$$

1-particle phase space:

$$dPS_1(p+q;p') = \frac{d^{n-1}p'}{(2\pi)^{n-1}2p'^0}(2\pi)^n\delta_n(p+q-p') = 2\pi d^n p'\delta_n(p+q-p')\delta_+(p'^2)$$

$$= 2\pi\delta\left[(p+q)^2\right] = \frac{2\pi}{Q^2}\delta(1-z)$$

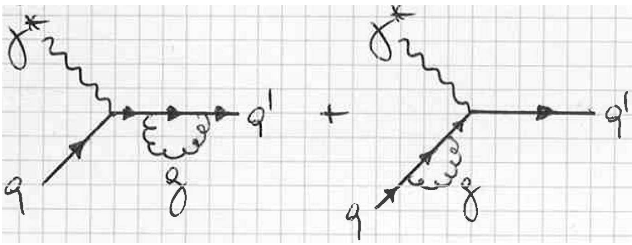
$$\hat{W}^{\mu\nu}_{LO} = \frac{e^2e_q^2}{\sigma_0}\ \frac{N_c}{N_c}\ Tr(\not{p}'\gamma^\mu\ \not{p}\gamma^\nu)\ \frac{1}{8\pi}\ \frac{2\pi}{Q^2}\ \delta(1-z)$$

$$\Rightarrow \boxed{\hat{\mathcal{F}}_{L,LO} = 0} \qquad \boxed{\hat{\mathcal{F}}_{1,LO} = \hat{\mathcal{F}}_{2,LO} = \delta(1-z)}$$

$$\sigma_0 = 2\pi\alpha e_q^2$$

2.) QCD corrections:

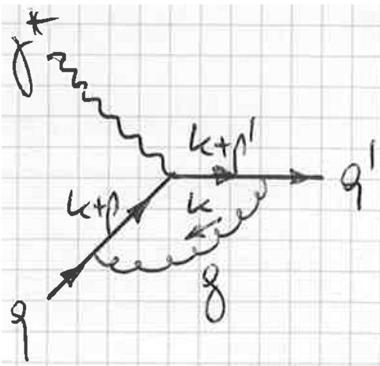
(i) virtual corrections:



$\equiv 0$ [partons massless]

since: $\displaystyle\int \frac{d^nk}{(2\pi)^n}\frac{1}{(k^2)^\alpha} = 0 \qquad \int \frac{d^nk}{(2\pi)^n}\frac{1}{k^2(k+p)^2} \propto \frac{(-p^2)^{-\epsilon}}{\epsilon} \to 0$

for $\epsilon < 0$ [analyt. cont.]



$$\mathcal{M}_V = i^6(-1)^4 e e_q \bar{g}_s^2 (T^aT^a)_{ij}\ *$$

$$*\ \bar{u}(p')\int \frac{d^nk}{(2\pi)^n}\frac{\gamma^\alpha(\not{k}+\not{p}')\gamma^\mu(\not{k}+\not{p})\gamma_\alpha}{k^2(k+p)^2(k+p')^2}\ u(p)$$

dimensionless coupling: $\bar{g}_s^2 = g_s^2\mu^{2\epsilon}$

$$\hat{W}^{\mu\nu}_V = \frac{1}{N_c}\sum 2\Re e\mathcal{M}^\mu_V\mathcal{M}^{*\nu}_{LO}\ \frac{dPS_1(p+q;p')}{8\pi\sigma_0}$$

$\Rightarrow \delta\widehat{\mathcal{F}}_{L,V} = 0$

$$\delta\widehat{\mathcal{F}}_{2,V} = \delta\widehat{\mathcal{F}}_{1,V} = i2C_F g_s^2 \left\{(3+2\epsilon)B_0(q;0,0) - 2Q^2 C_0(p,p';0,0,0)\right\}\delta(1-z)$$

$$B_0(q;0,0) = \mu^{2\epsilon}\int \frac{d^nk}{(2\pi)^n}\,\frac{1}{k^2(k+q)^2} = i\frac{\Gamma(\epsilon)}{(4\pi)^2}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\underbrace{\int_0^1 dx\; x^{-\epsilon}\,(1-x)^{-\epsilon}}_{= B(1-\epsilon,1-\epsilon) = \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}}$$

$$= i\frac{\Gamma(1+\epsilon)}{(4\pi)^2}\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\frac{1}{\epsilon(1-2\epsilon)}$$

Gamma function: $\Gamma(1+\epsilon) = \exp\left\{-\gamma_E\epsilon + \sum_{i=2}^{\infty}\frac{(-1)^i}{i}\zeta(i)\epsilon^i\right\}$

↑ Euler const. $\gamma_E = 0.577215\ldots$

↑ Riemann's Zeta fct. $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) = 1.202056\ldots$, $\zeta(4) = \frac{\pi^4}{90}$ etc.

$\Rightarrow \Gamma(1+\epsilon)\Gamma(1-\epsilon) = 1 + \epsilon^2\zeta(2) + \mathcal{O}(\epsilon^4)$

$$B_0(q;0,0) = \frac{i}{(4\pi)^2}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\left(\frac{1}{\epsilon}+2\right) + \mathcal{O}(\epsilon)$$

$$C_0(p,p';0,0,0) = \mu^{2\epsilon}\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{k^2(k+p)^2(k+p')^2}$$

$$= -i\frac{\Gamma(1+\epsilon)}{(4\pi)^2 Q^2}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\underbrace{\int_0^1 dxdy\; y^{-1-\epsilon}x^{-1-\epsilon}\,(1-x)^{-\epsilon}}$$

divergent for $\epsilon > 0$ ($n < 4$)

convergent for $\epsilon < 0$ ($n > 4$)

analytical continuation: $\int_0^1 dz\; z^{x-1}(1-z)^{y-1} \equiv B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Infrared and collinear sing. are regularized for $n > 4$ ($\epsilon < 0$) by analytical continuation.

$$\Rightarrow \int_0^1 dy\; y^{-1-\epsilon} = \frac{\Gamma(-\epsilon)\Gamma(1)}{\Gamma(1-\epsilon)} = -\frac{1}{\epsilon}$$

$$\int_0^1 dx\; x^{-1-\epsilon}\;(1-x)^{-\epsilon} = \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$C_0(p,p';0,0,0) = -i\frac{\Gamma(1+\epsilon)}{(4\pi)^2Q^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\frac{1}{\epsilon^2} \quad \leftarrow \text{IR, COLL}$$
$$= \frac{i}{(4\pi)^2Q^2}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\left[-\frac{1}{\epsilon^2} - \zeta(2)\right] + \mathcal{O}(\epsilon)$$

---

renormalization: $m = 0 \Rightarrow \delta m = 0$

$$\mathcal{M}^{\mu}_{CT} = \mathcal{M}^{\mu}_{LO}\left\{\sqrt{Z_2}^2\frac{\sqrt{Z_3}}{Z_1} - 1\right\}$$

Ward-identity: $Z_1 = Z_2$
photon propagator: $Z_3 = 1$

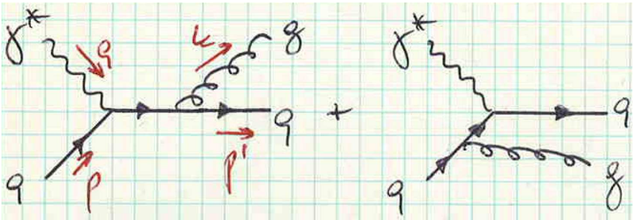
$\Rightarrow \mathcal{M}^{\mu}_{CT} = 0$ no renormalization after adding all diagrams

$$\boxed{\delta\hat{\mathcal{F}}_{2,V} = \delta\hat{\mathcal{F}}_{1,V} = C_F\frac{\alpha_s}{2\pi}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 - 2\zeta(2)\right]\delta(1-z)}$$

$$\boxed{\delta\hat{\mathcal{F}}_{L,V} = 0}$$

Infrared divergences will be subtracted by adding the real gluon radiation.

(ii) real corrections:



$$\left.\begin{array}{l} s = (p+q)^2 \\ t = (p'-p)^2 \\ u = (k-p)^2 \end{array}\right\} \Rightarrow s+t+u = q^2$$

$$\mathcal{M}_q^\mu = i^3(-1)^2 e e_q \bar{g}_s T_{ij}^a \bar{u}(p') \left\{ \frac{\gamma^\alpha(\not{p}'+\not{k})\gamma^\mu}{(p'+k)^2} + \frac{\gamma^\mu(\not{p}-\not{k})\gamma^\alpha}{(p-k)^2} \right\} u(p)\epsilon_\alpha^*$$

$$\widehat{W}_{\gamma q}^{\mu\nu} = \frac{1}{N_c} \int \sum \mathcal{M}_q^\mu \mathcal{M}_q^{*\nu} \frac{dPS_2(p+q;p',k)}{8\pi\sigma_0}$$

$$dPS_2(p+q;p',k) = \frac{d^{n-1}p'}{(2\pi)^{n-1}2p'^0} \frac{d^{n-1}k}{(2\pi)^{n-1}2k^0} (2\pi)^n \delta_n(p+q-p'-k)$$

$$= \frac{d^{n-1}k}{(2\pi)^{n-2}2k^0} d^n p' \delta_+(p'^2) \delta_n(p+q-p'-k)$$

$$= \frac{|\vec{k}|^{n-2} d|\vec{k}| d\Omega_{n-1}}{2(2\pi)^{n-2}k^0} \; \delta\left[(p+q-k)^2\right]$$

c.m.s.: $p+q = \sqrt{s}(1;\vec{0}) \Rightarrow (p+q-k)^2 = s - 2\sqrt{s}k^0$

$$0 = k^2 = (k^0)^2 - |\vec{k}|^2 \Rightarrow k^0 dk^0 = |\vec{k}| d|\vec{k}|$$

$$dPS_2 = \frac{(k^0)^{n-3} dk^0 d\Omega_{n-1}}{2(2\pi)^{n-2}} \; \delta(s - 2\sqrt{s}k^0) = \frac{s^{\frac{n-4}{2}}}{2^{2n-3}\pi^{n-2}} d\Omega_{n-1}$$

surface integral: θ = scattering angle of quark

$$d\Omega_{n-1} = \sin^{n-3}\theta d\theta d\Omega_{n-2} = 2\frac{\pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-2}{2}\right)} (1-\cos^2\theta)^{\frac{n-4}{2}} d\cos\theta$$

substitution: $y = \frac{1}{2}(1+\cos\theta)$

$$\boxed{dPS_2 = \frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^\epsilon \frac{y^{-\epsilon}(1-y)^{-\epsilon}}{\Gamma(1-\epsilon)} \; dy} \qquad 0 \le y \le 1$$

parametrization: $p = \frac{s+Q^2}{2\sqrt{s}}(1;0,\vec{0},1) \qquad q = \left(\frac{s-Q^2}{2\sqrt{s}};0,\vec{0},-\frac{s+Q^2}{2\sqrt{s}}\right)$

$$p' = \frac{\sqrt{s}}{2}(1;\sin\theta,\vec{0},\cos\theta) \quad k = \frac{\sqrt{s}}{2}(1;-\sin\theta,\vec{0},-\cos\theta)$$

$$\Rightarrow s = \frac{1-z}{z}Q^2; \quad t = -\frac{Q^2}{z}(1-y); \quad u = -\frac{Q^2}{z}y$$

$$\boxed{\delta\hat{\mathcal{F}}_{Lq} = \frac{4z^2}{Q^2}p_\mu p_\nu \hat{W}^{\mu\nu}_{\gamma q} = \int_0^1 dy\ \frac{4}{3}\frac{\alpha_s}{2\pi}4z^2\frac{-t}{Q^2} = \frac{4}{3}\ \frac{\alpha_s}{2\pi}\ 2z \neq 0} \leftarrow \text{finite}$$

$$\delta\hat{\mathcal{F}}_{2q} - \frac{3}{2}\delta\hat{\mathcal{F}}_{Lq} = \frac{-g_{\mu\nu}\hat{W}^{\mu\nu}_{\gamma q}}{2(1-\epsilon)} + \mathcal{O}(\epsilon)$$

$$= \frac{4}{3}\ \frac{\alpha_s}{2\pi}\ \frac{z^\epsilon(1-z)^{-\epsilon}}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon \int_0^1 dy\ y^{-\epsilon}(1-y)^{-\epsilon}\ *$$

$$* \quad \underbrace{\left\{(1-\epsilon)\left(\frac{s}{-u}+\frac{-u}{s}\right)+2\frac{tQ^2}{su}+2\epsilon\right\}}_{= (1-\epsilon)\left(\frac{1-z}{y}+\frac{y}{1-z}\right)+2\frac{(1-y)z}{y(1-z)}+2\epsilon}$$

$$= \frac{4}{3}\ \frac{\alpha_s}{2\pi}\ \frac{z^\epsilon(1-z)^{-\epsilon}}{\Gamma(1-\epsilon)}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon *$$

$$*\left\{(1-\epsilon)\left[\frac{1-z}{-\epsilon}+\frac{1-\epsilon}{(2-2\epsilon)(1-2\epsilon)(1-z)}\right]-2\frac{(1-\epsilon)}{\epsilon(1-2\epsilon)}\ \frac{z}{1-z}+\frac{2\epsilon}{1-2\epsilon}\right\}$$

distributions:

$$(1-z)^{-1-\epsilon} = \left(\frac{1}{(1-z)^{1+\epsilon}}\right)_+ + \delta(1-z)\int_0^1 \frac{dz'}{(1-z')^{1+\epsilon}}$$

$$= -\frac{1}{\epsilon}\delta(1-z)+\left(\frac{1}{1-z}\right)_+ - \epsilon\left(\frac{\log(1-z)}{1-z}\right)_+ + \mathcal{O}(\epsilon^2)$$

$$\int_a^1 dz\ \frac{f(z)}{(1-z)_+} = \int_a^1 dz\ \frac{f(z)-f(1)}{1-z} - \int_0^a \frac{dz}{1-z}\ f(1)$$

$$\delta\widehat{\mathcal{F}}_{2q} - \frac{3}{2}\delta\widehat{\mathcal{F}}_{Lq} = C_F\ \frac{\alpha_s}{2\pi}\ \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\left\{\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{7}{2}\right]\delta(1-z)\right.$$

$$\left. -\left(\frac{1}{\epsilon}+\log z\right)\left(\frac{1+z^2}{1-z}\right)_+ + (1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+ - \frac{3}{2}\left(\frac{1}{1-z}\right)_+ + 3 - z\right\}$$

Altarelli–Parisi splitting functions:

$$P_{qq}(z) = C_F\frac{1+z^2}{1-z} - \delta(1-z)\int_0^1 dz'\ C_F\ \frac{1+z'^2}{1-z'}$$

$$\Rightarrow \int_0^1 dz\ f(z)\ P_{qq}(z) = \int_0^1 dz\ C_F\frac{1+z^2}{1-z}\left[f(z)-f(1)\right] = \int_0^1 dz\ C_F\left(\frac{1+z^2}{1-z}\right)_+ f(z)$$

$$\Rightarrow \boxed{P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+ = C_F\left\{\left(\frac{2}{1-z}\right)_+ - 1 - z + \frac{3}{2}\delta(1-z)\right\}}$$

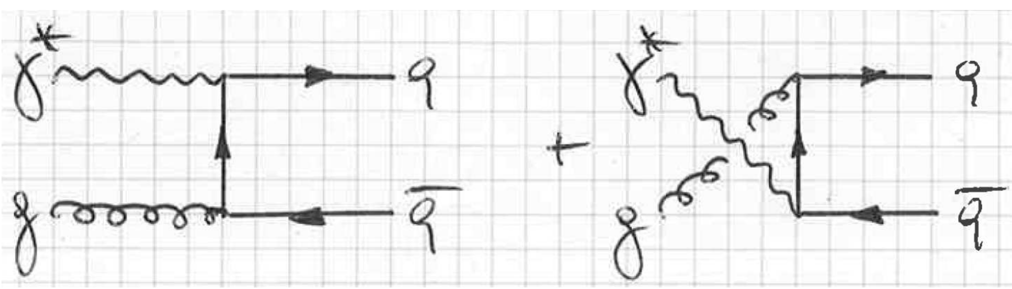
sum virtual + real:

$$\delta\widehat{\mathcal{F}}_2^{\gamma q} = \widehat{\mathcal{F}}_{2V} + \widehat{\mathcal{F}}_{2q}$$

$$= \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\frac{\alpha_s}{2\pi}\left[-\frac{1}{\epsilon}-\log z\right]P_{qq}(z) + C_F\ \frac{\alpha_s}{2\pi}\ \left\{(1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+\right.$$

$$\left. -\frac{3}{2}\left(\frac{1}{1-z}\right)_+ + 3 + 2z - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\right\}$$

The crossed channel $\gamma^* g \to q\bar{q}$ is of the same order in α_s and cannot be distinguished from $\gamma^* \to qg$.



$$\delta\hat{\mathcal{F}}_L^{\gamma g} = T_R\ \frac{\alpha_s}{2\pi}\ 4z(1-z) \qquad\qquad \left[T_R = \frac{1}{2}\right]$$

$$\delta\hat{\mathcal{F}}_2^{\gamma g} = \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\left[-\frac{1}{\epsilon} - \log\frac{z}{1-z}\right]\frac{\alpha_s}{2\pi}\ P_{qg}(z) + T_R\ \frac{\alpha_s}{2\pi}\ [8z(1-z)-1]$$

$$P_{qg}(z) = T_R[z^2 + (1-z)^2]$$

gluon-spin average: $\frac{1}{2} \to \frac{1}{n-2} = \frac{1}{2(1-\epsilon)}$

DIS structure function:

$$F_2(x,Q^2) = 2x\sum_q e_q^2\left\{[q_0 + \bar{q}_0]\otimes\hat{\mathcal{F}}_2^{\gamma q} + g_0\otimes\hat{\mathcal{F}}_2^{\gamma g}\right\}$$

with the convolution:

$$f\otimes g = \int_0^1 dydz\ f(y)\ g(z)\ \delta(x-yz) = \int_x^1\frac{dz}{z}\ f\left(\frac{x}{z}\right)g(z)$$

The remaining collinear singularities are removed by the renormalization of the parton densities.

Renormalization of the parton densities [mass factorization]:

$$q_0(x) = F_{qq}\otimes q(x,\mu_F^2) + F_{qg}\otimes g(x,\mu_F^2)$$

$$F_{ij}(x) = \delta_{ij}\delta(1-x) + \frac{\alpha_s}{2\pi}\left\{\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}P_{ij}(x) - f_{ij}(x)\right\}$$

μ_F = factorization scale of the parton densities

$$\Rightarrow \mu_F^2\frac{\partial q_0(x)}{\partial\mu_F^2} = 0 = -\frac{\alpha_s}{2\pi}\left[P_{qq}\otimes q(x,\mu_F^2) + P_{qg}\otimes g(x,\mu_F^2)\right] + \mu_F^2\frac{\partial q(x,\mu_F^2)}{\partial\mu_F^2} + \mathcal{O}(\alpha_s^2)$$

$\Rightarrow q(x,\mu_F^2)$ is solution of the Alterelli–Parisi equations at LO.

Result:

$$F_2(x,Q^2) = 2x\sum_q e_q^2[q(x,\mu_F^2)+\bar{q}(x,\mu_F^2)] + \Delta F_2(x,Q^2)$$

$$\Delta F_2(x,Q^2) = 2x\frac{\alpha_s}{2\pi}\sum_q e_q^2\int_x^1 \frac{dz}{z}\left\{C_F\left[q\left(\frac{x}{z},\mu_F^2\right)+\bar{q}\left(\frac{x}{z},\mu_F^2\right)\right] * \right.$$

$$* \left[-\frac{P_{qq}(z)}{C_F}\log\frac{\mu_F^2 z}{Q^2} + (1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+ - \frac{3}{2}\left(\frac{1}{1-z}\right)_+ + 3 + 2z\right.$$

$$\left. - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z) - \frac{f_{qq}(z)}{C_F}\right]$$

$$\left. + T_R\ g\left(\frac{x}{z},\mu_F^2\right)\left[-\frac{P_{qg}(z)}{T_R}\log\frac{\mu_F^2 z}{Q^2(1-z)} + 8z(1-z) - 1 - \frac{f_{qg}(z)}{T_R}\right]\right\}$$

$$F_L(x,Q^2) = F_2(x,Q^2) - 2x\ F_1(x,Q^2)$$

$$= 2x\frac{\alpha_s}{2\pi}\sum_q e_q^2\int_x^1\frac{dz}{z}\left\{C_F\left[q\left(\frac{x}{z},\mu_F^2\right)+\bar{q}\left(\frac{x}{z},\mu_F^2\right)\right]2z + T_R\ g\left(\frac{x}{z},\mu_F^2\right)4z(1-z)\right\}$$

PHYSICAL INTERPRETATION:

1.) natural factorization scale: $\mu_F^2 = Q^2$

2.) factorization scheme:

(i) $\overline{MS}$ scheme: $f_{ij}^{\overline{MS}}(z) \equiv 0$

(ii) DIS scheme: $F_2(x,Q^2) \equiv 2x\sum_q e_q^2\left[q(x,Q^2)+\bar{q}(x,Q^2)\right]$

$$\Rightarrow \Delta F_2 \equiv 0 \quad [\mu_F^2 = Q^2]$$

$$\Rightarrow f_{qq}^{DIS}(z) = C_F\left\{-\frac{1+z^2}{1-z}\log z + (1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+ - \frac{3}{2}\left(\frac{1}{1-z}\right)_+ \right.$$

$$\left. + 3 + 2z - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\right\}$$

$$f_{qg}^{DIS}(z) = T_R\left\{[z^2+(1-z)^2]\log\frac{1-z}{z} + 8z(1-z) - 1\right\}$$

FACTORIZATION THEOREM OF QCD

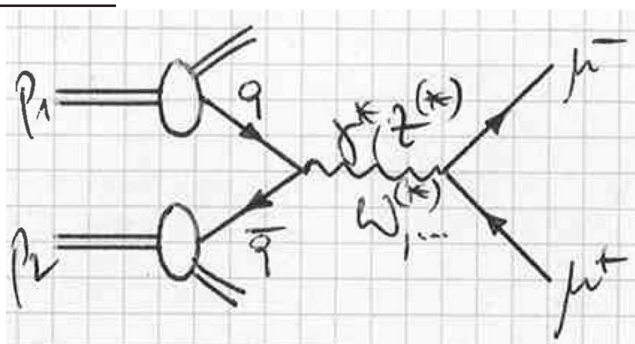
Partonic cross sections develop collinear divergencies in the hadronic initial state that factorize universally [process-independent] from the hard scattering process and can be absorbed in the renormalized parton densities of the initial state. These renormalized parton densities are solutions of the DGLAP equations.

§5. Drell–Yan Processes

Production of elw. int. particles in hadron coll.

$p\bar{p} \to \mu^+\mu^- + X$

$p\bar{p} \to W^\pm, Z + X \quad \ldots$



cxn: $\sigma(p_1 p_2 \to \mu^+\mu^- + X)$

$$= \sum_q \int_0^1 dx_1 dx_2 [q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)] \hat{\sigma}(q\bar{q} \to \mu^+\mu^-; \hat{s})$$

invariant energy: $\hat{s} = (p_q + p_{\bar{q}})^2 = 2p_q p_{\bar{q}} = x_1 x_2 * 2p_1 p_2 = x_1 x_2 s$

parton cxn: $\hat{\sigma}(q\bar{q} \to \mu^+\mu^-) = \left(\frac{1}{N_c}\right)^2 N_c \frac{4\pi\alpha^2}{2\hat{s}} e_q^2 = \frac{e_q^2}{N_c} \frac{4\pi\alpha^2}{2\hat{s}}$

integration boundaries: $\hat{s} = x_1 x_2 s \geq (2m_\mu)^2$

$$\Rightarrow x_2 \geq \frac{\tau_0}{x_1}; \quad x_1 \geq \tau_0 = \frac{4m_\mu^2}{s}$$

$$\sigma = \int_{\tau_0}^1 d\tau \left[\sum_q \int_{\tau_0}^1 dx_1 \int_{\frac{\tau_0}{x_1}}^1 dx_2 [q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)] \delta_1(\tau - x_1 x_2) \right] \hat{\sigma}(\hat{s} = \tau s)$$

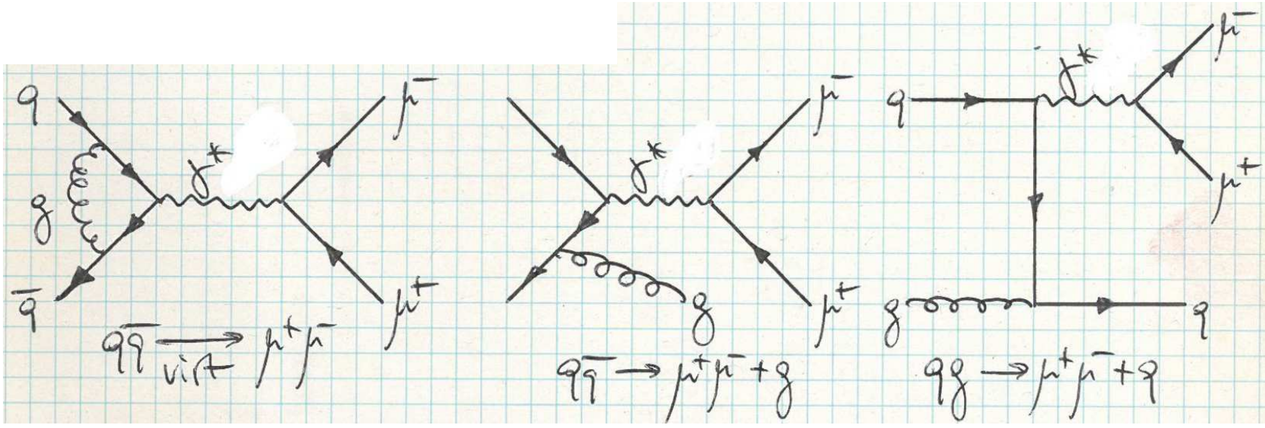
$$= \int_{\tau_0}^1 d\tau \sum_q \int_\tau^1 \frac{dx}{x} \left[q(x)\bar{q}\left(\frac{\tau}{x}\right) + \bar{q}(x) q\left(\frac{\tau}{x}\right) \right] \hat{\sigma}(\tau s)$$

$$\sigma = \int_{\tau_0}^1 d\tau \sum_q \frac{d\mathcal{L}^{q\bar{q}}}{d\tau} \; \hat{\sigma}(\tau s)$$

$\uparrow$ "luminosity of $q, \bar{q}$ in hadron beams"

$$M_{\mu\mu}^4 \frac{d\sigma}{dM_{\mu\mu}^2} = \frac{4\pi\alpha^2}{3N_c} \tau_\mu \sum_q e_q^2 \left. \frac{d\mathcal{L}^{q\bar{q}}}{d\tau} \right|_{\tau_\mu = \frac{M_{\mu\mu}^2}{s}}$$

<u>QCD corrections:</u>



$$M_{\mu\mu}^4 \frac{d\sigma}{dM_{\mu\mu}^2} = \frac{4\pi\alpha^2}{3N_c} \tau_\mu \int_{\tau_0}^1 \frac{d\tau}{\tau} \left\{ \sum_q e_q^2 \frac{d\mathcal{L}^{q\bar{q}}}{d\tau} \left[\delta(1-z) + \frac{\alpha_s(\mu_R^2)}{\pi} D_{qq}(z) \right] + \sum_{q,\bar{q}} e_q^2 \frac{d\mathcal{L}^{gq}}{d\tau} \; \frac{\alpha_s(\mu_R^2)}{\pi} D_{gq}(z) \right\} \qquad \left[z = \frac{\tau_\mu}{\tau} \right]$$

$$D_{qq}(z) = -P_{qq}(z) \log \frac{\mu_F^2 z}{M_{\mu\mu}^2} + C_F \left\{ 2 \left[\frac{\pi^2}{6} - 2 \right] \delta(1-z) + 2(1+z^2) \left(\frac{\log(1-z)}{1-z} \right)_+ \right\} - f_{qq}(z)$$

$$D_{gq}(z) = -\frac{1}{2} P_{qg}(z) \log \frac{\mu_F^2 z}{M_{\mu\mu}^2 (1-z)^2} + \frac{T_R}{4} (1 + 6z - 7z^2) - f_{qg}(z)$$

$$\frac{d\mathcal{L}^{gq}}{d\tau} = \int_\tau^1 \frac{dx}{x} \left[q(x, \mu_F^2) g\left(\frac{\tau}{x}, \mu_F^2\right) + g(x, \mu_F^2) q\left(\frac{\tau}{x}, \mu_F^2\right) \right]$$

Remark: In complete analogy the following processes can be treated due to the factorization theorem:

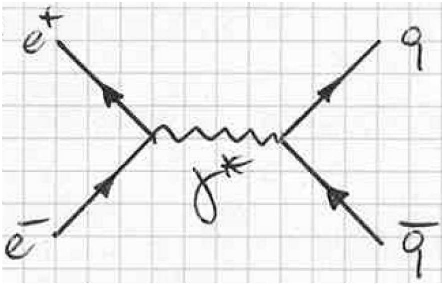
$p_1 p_2 \to W^\pm, Z + X$

$p_1 p_2 \to \tilde{\ell}\bar{\tilde{\ell}} + X$

$p_1 p_2 \to nj + X$

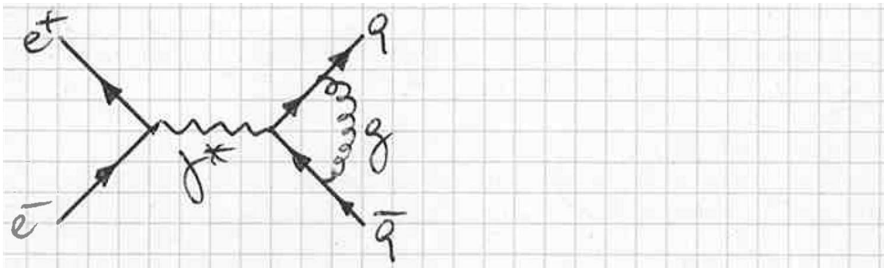
§6. $e^+e^- \to$ hadrons: total cxn

parton picture:

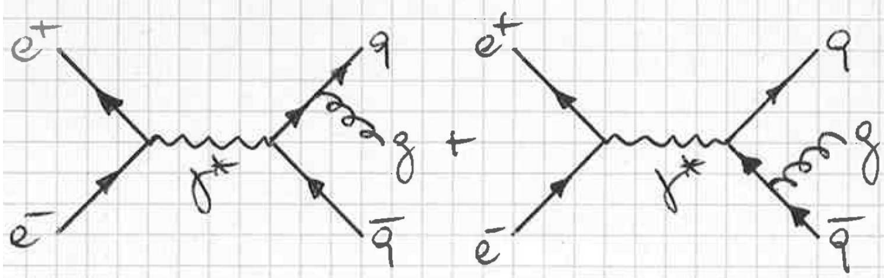


$$\sigma_0 = \frac{4\pi\alpha^2}{3s}\ N_c e_q^2$$

QCD: virt. corr.



real corr.



$$\begin{aligned}
\sigma &= (1 + \delta_V + \delta_R)\ \sigma_0 \\
\delta_V &= C_F \frac{\alpha_s}{\pi} \Gamma(1+\epsilon) \left(\frac{4\pi\mu^2}{s}\right)^\epsilon \left\{ -\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} - 4 + \frac{2}{3}\pi^2 + \mathcal{O}(\epsilon) \right\} \\
\delta_R &= C_F \frac{\alpha_s}{\pi} \Gamma(1+\epsilon) \left(\frac{4\pi\mu^2}{s}\right)^\epsilon \left\{ \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{19}{4} - \frac{2}{3}\pi^2 + \mathcal{O}(\epsilon) \right\}
\end{aligned}$$

total cxn: $\sigma = \left(1 + \frac{3}{4} C_F \frac{\alpha_s}{\pi}\right) \sigma_0 = \left(1 + \frac{\alpha_s}{\pi}\right) \sigma_0$

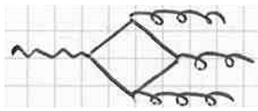
$$R\text{-value} = \frac{\sigma(e^+e^- \to had)}{\sigma(e^+e^- \to \mu^+\mu^-)} : \qquad R_B = 3\sum_q e_q^2$$

$$R = R_B\left[1 + \frac{\alpha_s}{\pi} + \sum_{n\geq 2} c_n \left(\frac{\alpha_s}{\pi}\right)^n\right] \qquad \text{for } \overline{MS} \text{ @ } \mu^2 = s$$

$$c_2 = \frac{365}{24} - 11\zeta(3) - \left(\frac{11}{12} - \frac{2}{3}\zeta(3)\right) N_F \approx 1.986 - 0.115 N_F$$

$$c_3 = \frac{87029}{288} - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)$$
$$- \left(\frac{7847}{216} - \frac{262}{9}\zeta(3) + \frac{25}{9}\zeta(5)\right) N_F$$
$$+ \left(\frac{151}{162} - \frac{19}{27}\zeta(3)\right) N_F^2$$
$$- \frac{1}{72}\zeta(2)(33 - 2N_F)^2 + \eta\left(\frac{55}{72} - \frac{5}{3}\zeta(3)\right)$$
$$\approx -6.637 - 1.200 N_F - 0.005 N_F^2 - 1.240\eta$$

$$\eta = \frac{(\sum e_q)^2}{3\sum e_q^2}$$



high precision determination of $\alpha_s(M_Z^2)^{\overline{MS}}_{(5)} = 0.122 \pm 0.003$

§7. Jets in QCD

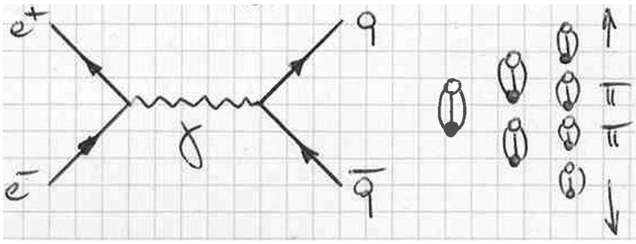
asympt. freedom: In the femto-universe $d \lesssim 10^{-15}$ cm strongly interacting processes proceed as one quantum processes on the level of quarks and gluons. [$\to$ analogous to e and γ in QED]

Jet hypothesis: Parton configurations built up in the femto-universe transform at large distances $d \gtrsim 10^{-13}$ cm into bundles of hadrons with limited transverse momentum $p_\perp \lesssim 500$ MeV $\equiv$ jets

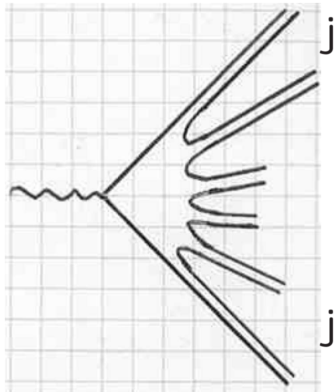
$\Rightarrow$ jet analyses: tests of QCD in femto-universe

jet structure: determined by (non-)perturbative QCD

(a) 0^{th} order QCD: $e^+e^- \to q\bar{q}$

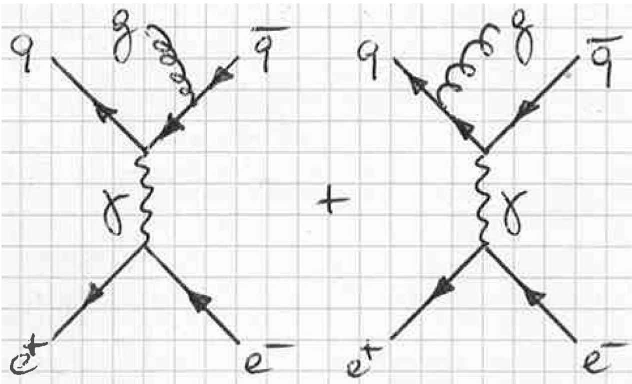


energy flux tube: spont. $q\bar{q}$ production $\Rightarrow$ break up of flux tube with small $p_\perp$



"SPEAR"-Jets

(b) gluon jets in e^+e^- annihilation:



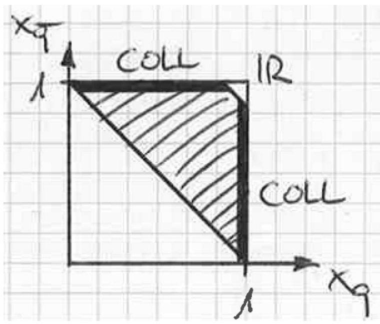
acceleration of color charge $\Rightarrow$ radiation of gluonic gauge quanta [$\sim \gamma$ radiation off accelerated charges]

$q\bar{q}$ kinematics: $x_i = \dfrac{E_i}{\sqrt{s}/2}$

with $x_q + x_{\bar{q}} + x_g = 2$

$0 \le x_i \le 1$

$x_q + x_{\bar{q}} = 2 - x_g \ge 1$



Dalitz plot

pole for $(q+g)^2 = (Q-\bar{q})^2 = Q^2 - 2Q\bar{q} = Q^2(1-x_{\bar{q}})$

$(\bar{q}+g)^2 = Q^2(1-x_q) \qquad [Q = e^+ + e^- = \sqrt{s}(1;\vec{0})]$

$$\frac{1}{\sigma_{q\bar{q}}}\frac{d\sigma}{dx_q dx_{\bar{q}}} = \frac{2}{3}\,\frac{\alpha_s}{\pi}\,\frac{x_q^2 + x_{\bar{q}}^2}{(1-x_q)(1-x_{\bar{q}})}$$

$x_q \to 1 : g \parallel \bar{q}$ coll. conf.
$x_{\bar{q}} \to 1 : g \parallel q$
$x_q, x_{\bar{q}} \to 1 : x_g \to 0$ infrared

Exp. development: Increase of energy $\Rightarrow$
- jets become broader
- clear 3-jet events: PETRA-jets

$\leftarrow$ visible QCD gauge quanta

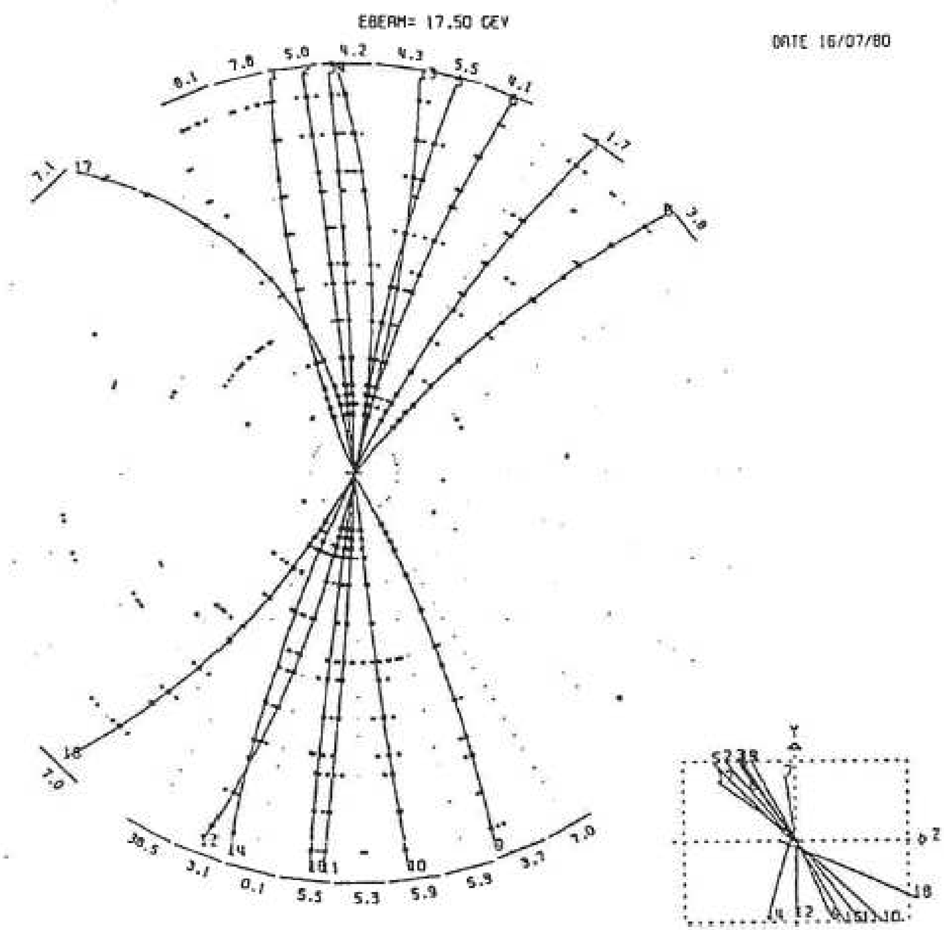


Fig. 29 A two-jet events as observed at W = 35 GeV in the TASSO detector.

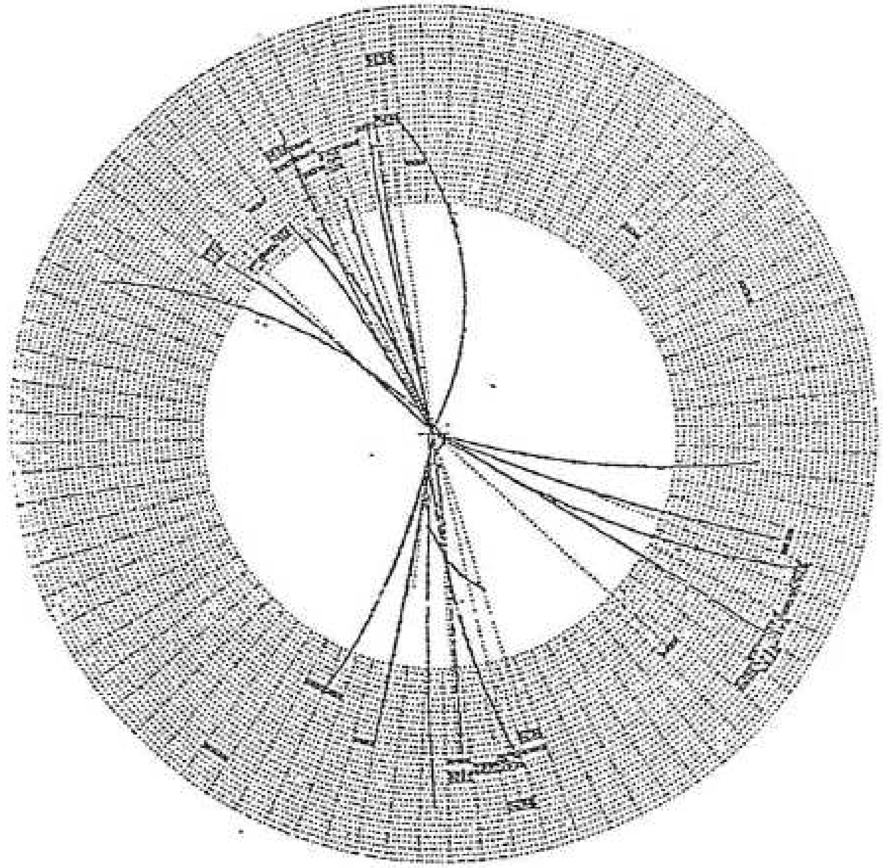


Fig. 11.12 A three-jet event observed by the JADE detector at PETRA.

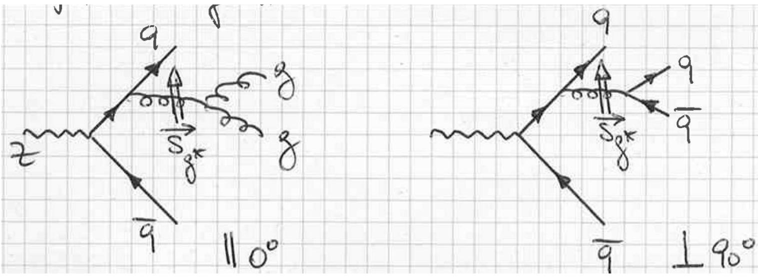
- measurement of gluon spin:

$$s_g = 1 \Rightarrow \rho_1 \sim \frac{1}{(1-x_q)(1-x_{\bar{q}})} \qquad \text{div. for } x_g \to 0; x_q, x_{\bar{q}} \to 1$$

$$s_g = 0 \Rightarrow \rho_0 \sim \frac{x_g^2}{(1-x_q)(1-x_{\bar{q}})} \qquad \text{finite for } x_g \to 0$$

- measurement of gluon color:
  4-jet events:

$$\chi = \angle(E_{12}, E_{34}) :$$



$$gg = \frac{(1-z+z^2)^2}{z(1-z)} + z(1-z)\cos 2\chi$$

$$q'\bar{q}' = \frac{1}{2}[z^2 + (1-z)^2] - z(1-z)\cos 2\chi$$

SU_3 : 0° vs. U_1 : 90°

- jet multiplicity:

$f_n(y) =$ fraction of events with n jets in final state:

$$\sum f_n(y) = 1$$

$y =$ max. jet mass: $M_{jet}^2 \leq ys$

$$f_{n+2}(y) = \left(\frac{\alpha_s}{2\pi}\right)^n \sum_{j=0}^{\infty} C_{nj}(y) \left(\frac{\alpha_s}{2\pi}\right)^j \quad \Rightarrow \text{measurement of } \alpha_s$$

Ex.: 2- and 3-jet distributions:

$$f_3 = \int_{(p_i+p_j)^2 \geq ys} dx_1 dx_2 \, \frac{2}{3} \, \frac{\alpha_s}{\pi} \, \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

$$= \frac{2}{3} \, \frac{\alpha_s}{\pi} \left[(3-6y)\log\frac{y}{1-2y} + 2\log^2\frac{y}{1-y} + \frac{5}{2} - 6y - \frac{9}{2}y^2 \right.$$

$$\left. + 4Li_2\left(\frac{y}{1-y}\right) - \frac{\pi^2}{3} \right]$$

$$f_2 = 1 - f_3 \qquad Li_2(x) = -\int_0^x \frac{dy}{y}\log(1-y) = \sum \frac{x^n}{n^2} \text{ for } |x| \leq 1$$

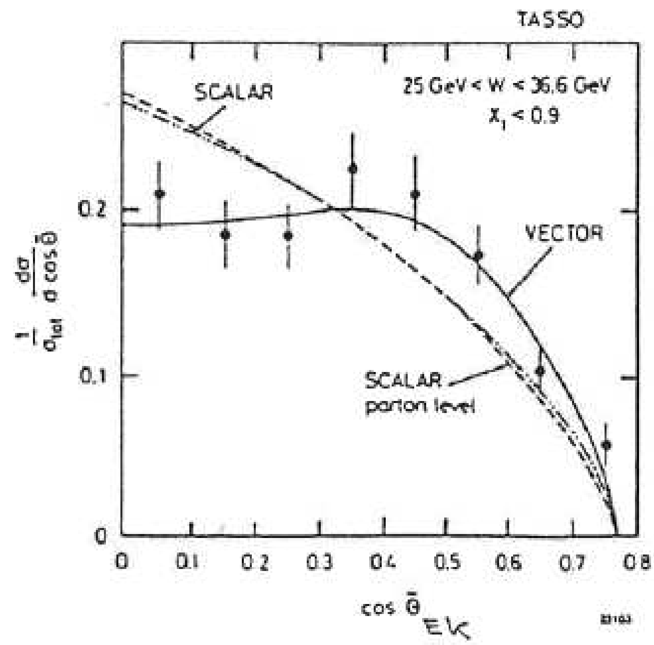


Fig. 7.25

The cosθ distribution of events with x_1 < 0.9. The solid line and the dashed-dotted line show the distribution predicted for vector gluons and scalar gluons respectively. The predictions include hadronization. For comparison the prediction for a scalar gluon on the parton level is shown in Fig. 7.24. The distributions are normalized to the number of observed events. EK = Ellis - Karliner

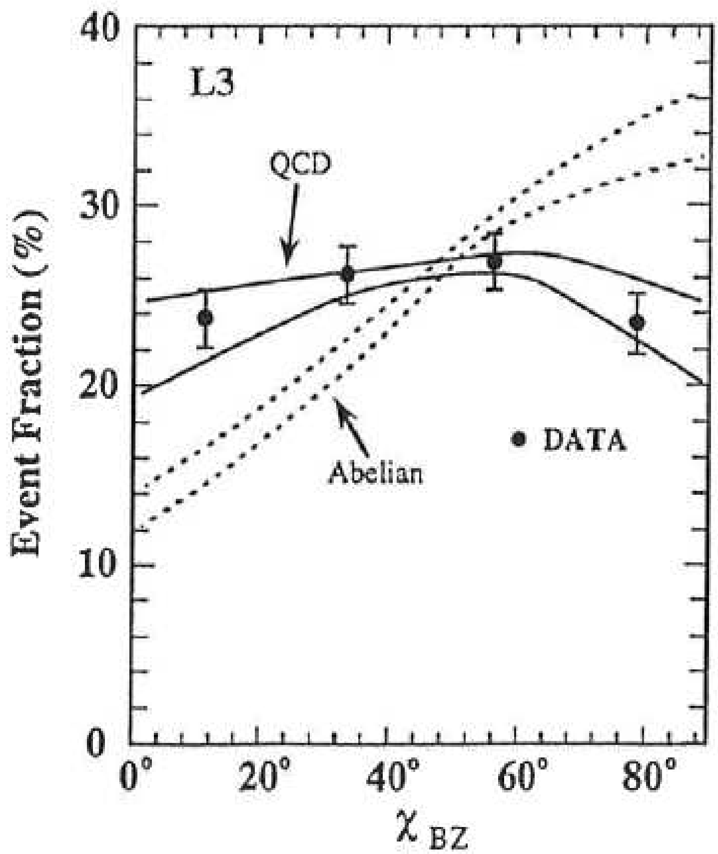
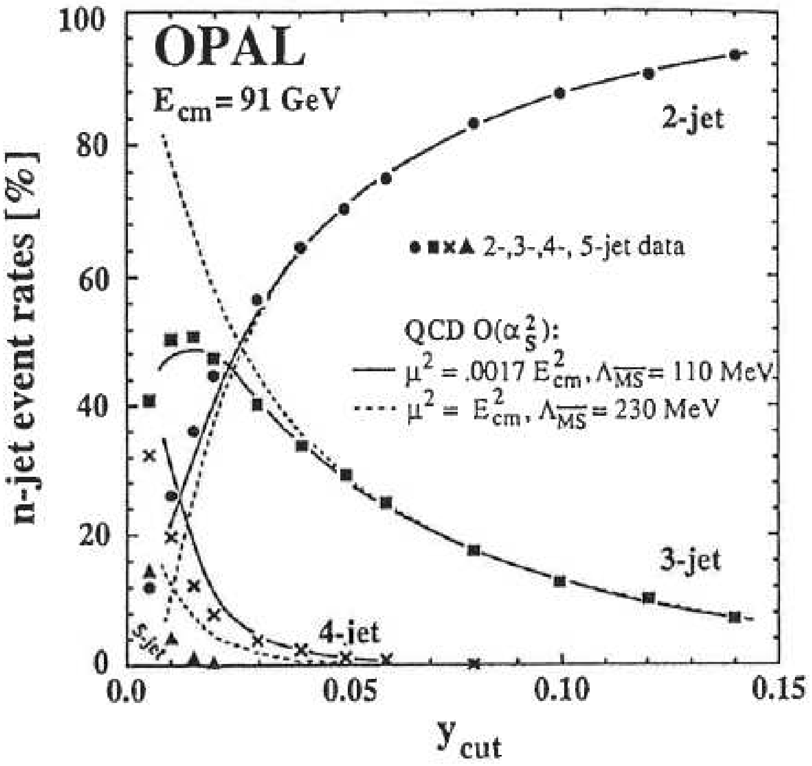


Fig. 3.11. Distribution in the Bengtsson-Zerwas angle at LEP. Figure from ref. [26].



.7. QCD fits to the jet rates at LEP, as measured by the OPAL collabo-
Figure from ref. [15].

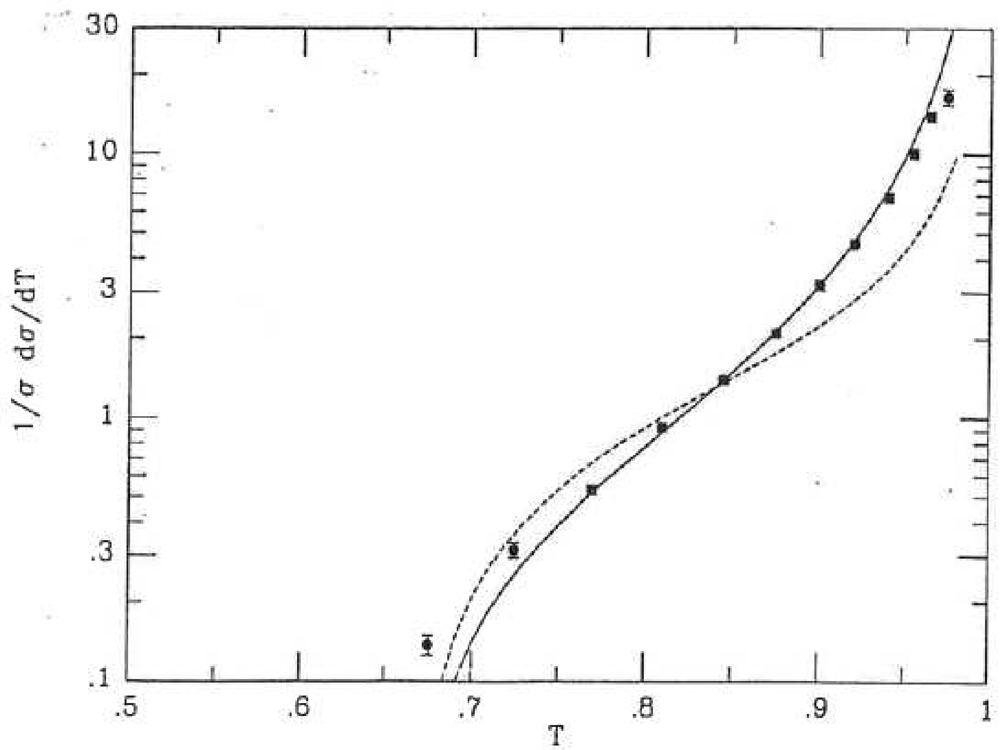
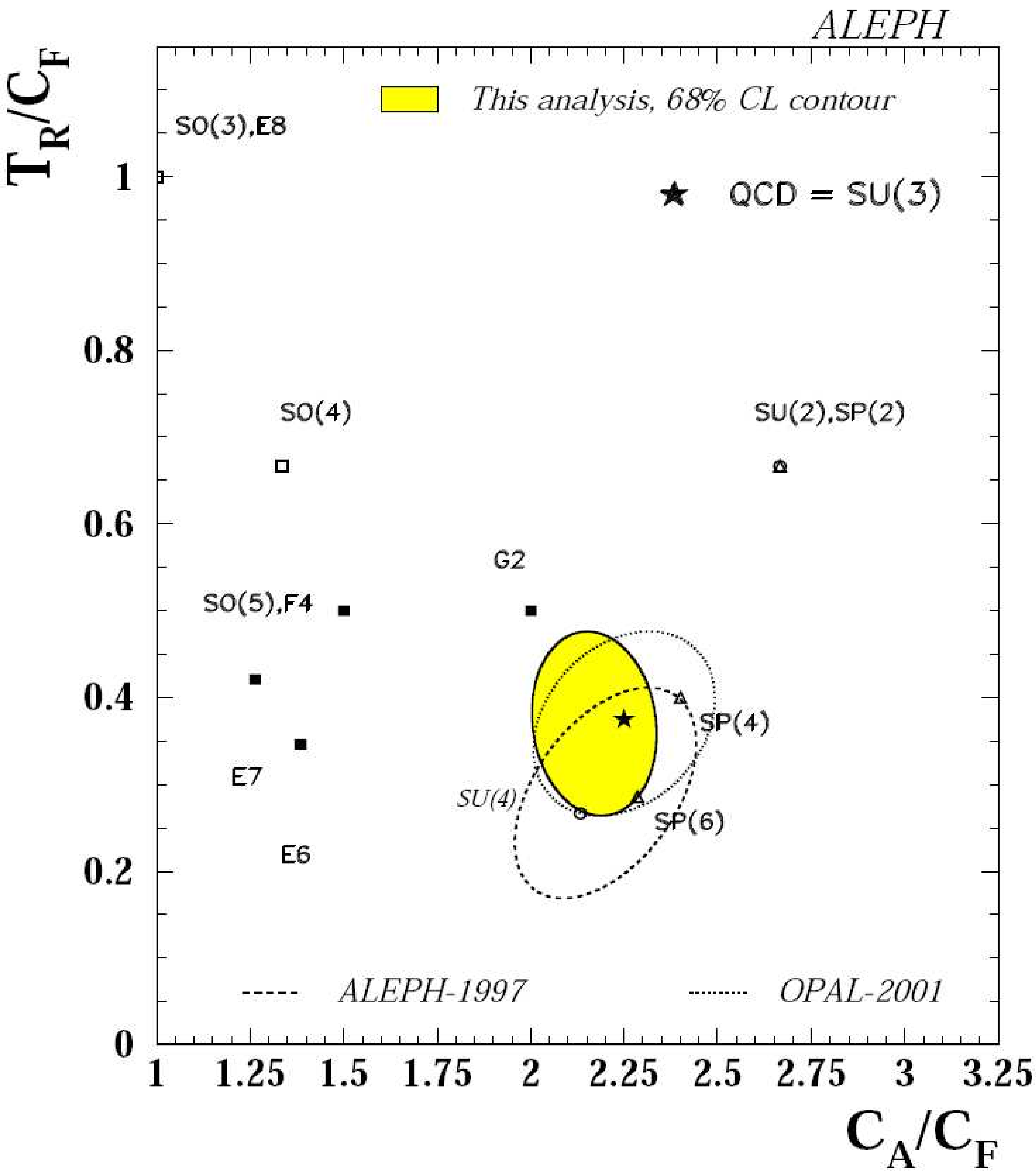
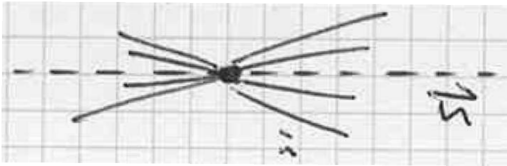


Fig. 3.9. The thrust distribution measured at LEP, showing data from the DELPHI collaboration [22] for $T < 0.98$, together with predictions of scalar gluon (dashed line) and vector gluon (solid line) theories.


ALEPH
This analysis, 68% CL contour
QCD = SU(3)
SO(3),E8
SO(4)
SU(2),SP(2)
G2
SO(5),F4
SP(4)
E7
SU(4)
SP(6)
E6
ALEPH-1997
OPAL-2001
T_R/C_F
C_A/C_F
0
0.2
0.4
0.6
0.8
1
1
1.25
1.5
1.75
2
2.25
2.5
2.75
3
3.25

• Shape variables: thrust, sphericity, masses, C-parameter,. . .

thrust: $T = \max_{\vec{n}} \frac{\sum |\vec{p}_i \vec{n}|}{\sum |\vec{p}_i|}$ $\quad 3j: \ \frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{2}{3}\frac{\alpha_s}{\pi}\left\{\frac{2(3T^2-3T+2)}{T(1-T)}\log\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right\}$

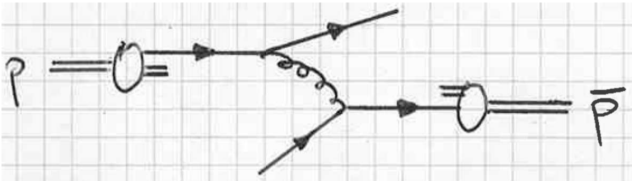


2 jets: $T = 1$

spher.: $T = \frac{1}{2}$

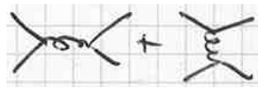
$q\bar{q}g$: $T = \max x_i$

(c) jets in high-energy $p\bar{p}$ scattering at large transv. mom.



large $p_\perp$ in subsystem $\Rightarrow$ small space-time distance for scattering process

Rutherford process: $q+q \to q+q, q+\bar{q} \to q+\bar{q}$

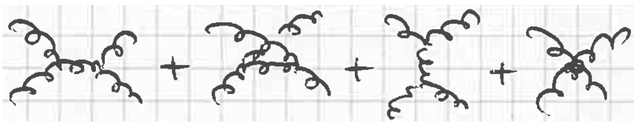


Compton scattering: $g+q \to g+q, g+\bar{q} \to g+\bar{q}$

annihilation: $q+\bar{q} \to g+g$

gluon fusion: $g+g \to q+\bar{q}$

gluon scattering: $g+g \to g+g$



• convolution with parton densities:

$$\frac{d\sigma}{dA}(p\bar{p} \to j_1 j_2 + \cdots) = \sum_{q,g} \int_0^1 dx_1 f_1(x_1, Q^2) \int_0^1 dx_2 f_2(x_2, Q^2) \int d\hat{\sigma}(p_1 p_2 \to p_1' p_2'; \hat{s} = x_1 x_2 s)\ \delta[A - A(p_i)]$$

• detection of Rutherford scattering in quark-gluon sector:

$$\frac{d\sigma^R}{d\cos\theta} \sim \frac{1}{\sin^4\frac{\theta}{2}} \sim \frac{1}{(1-\cos\theta)^2} \text{ strongly increasing for } \theta \to 0$$

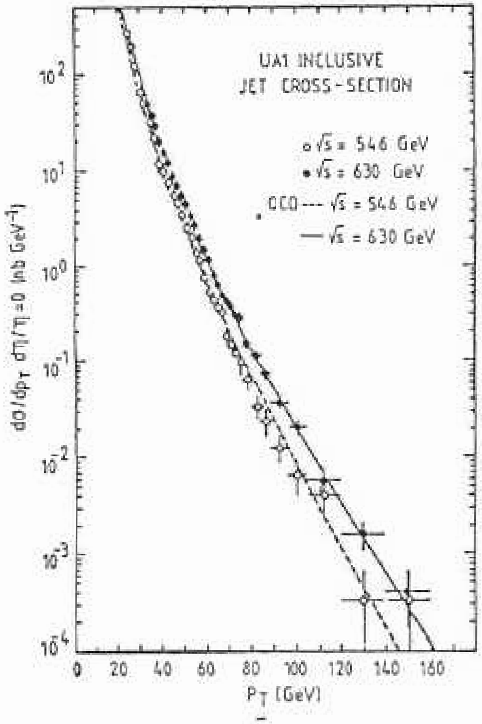
$$\chi = \frac{1+\cos\theta}{1-\cos\theta} \quad d\chi \sim \frac{d\cos\theta}{(1-\cos\theta)^2} \Rightarrow \frac{d\sigma^R}{d\chi} = \text{flat}$$

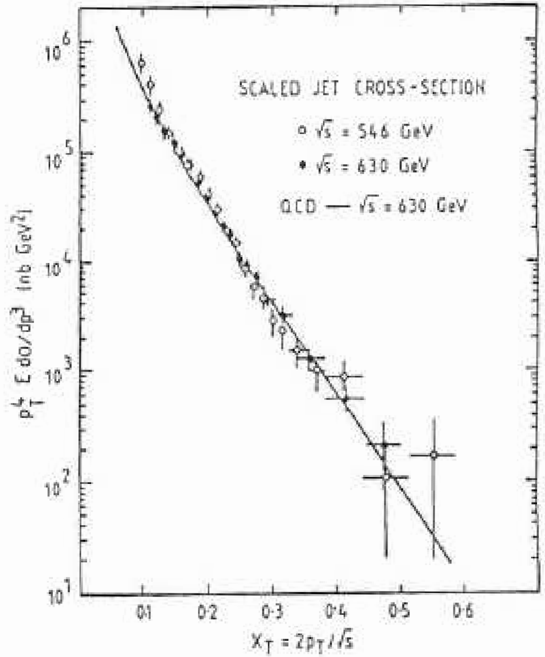
modulo: χ-dependence in $d\sigma^R$

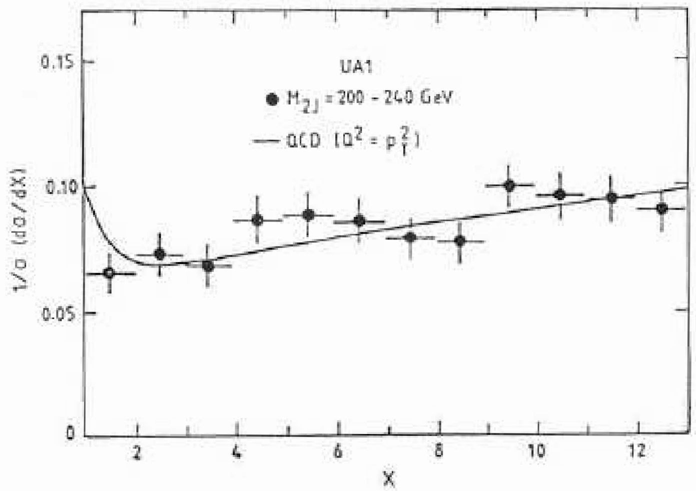
Q^2-dependence in $\alpha_s(Q^2)$, quark densities

Tabelle 20-1 Die Querschnitte für die möglichen Parton-Parton-2-Teilchen-Reaktionen in führender Ordnung in α_s. Dabei numerieren wir mit i, j $(1 \leq i,\ j \leq f)$ die verschiedenen Quark-Flavors durch (s. Gl. (19-2)).

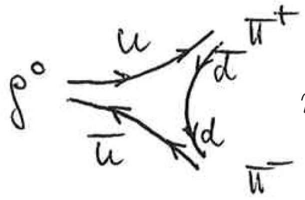
Reaktion	Streuquerschnitt $\frac{\hat{s}^2}{\pi \alpha_s^2} \frac{d\sigma}{d\hat{t}}$
$q^i + q^j \longrightarrow q^i + q^j \ (i \neq j)$ $q^i + \bar{q}^j \longrightarrow q^i + \bar{q}^j \ (i \neq j)$	$\frac{4}{9} \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$
$q^i + q^i \longrightarrow q^i + q^i$	$\frac{4}{9}\left(\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} + \frac{\hat{s}^2 + \hat{t}^2}{\hat{u}^2}\right) - \frac{8}{27} \frac{\hat{s}^2}{\hat{u}\hat{t}}$
$q^i + \bar{q}^i \longrightarrow q^j + \bar{q}^j \ (i \neq j)$	$\frac{4}{9} \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2}$
$q^i + \bar{q}^i \longrightarrow q^i + \bar{q}^i$	$\frac{4}{9}\left(\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} + \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2}\right) - \frac{8}{27} \frac{\hat{u}^2}{\hat{s}\hat{t}}$
$q^i + \bar{q}^i \longrightarrow G + G$	$\frac{32}{27} \frac{\hat{u}^2 + \hat{t}^2}{\hat{u}\hat{t}} - \frac{8}{3} \frac{\hat{u}^2 + \hat{t}^2}{\hat{s}^2}$
$G + G \longrightarrow q^i + \bar{q}^i$	$\frac{1}{6} \frac{\hat{u}^2 + \hat{t}^2}{\hat{u}\hat{t}} - \frac{3}{8} \frac{\hat{u}^2 + \hat{t}^2}{\hat{s}^2}$
$q^i + G \longrightarrow q^i + G$ $\bar{q}^i + G \longrightarrow \bar{q}^i + G$	$-\frac{4}{9} \frac{\hat{u}^2 + \hat{s}^2}{\hat{u}\hat{s}} + \frac{\hat{u}^2 + \hat{s}^2}{\hat{t}^2}$
$G + G \longrightarrow G + G$	$\frac{9}{2}\left(3 - \frac{\hat{u}\hat{t}}{\hat{s}^2} - \frac{\hat{u}\hat{s}}{\hat{t}^2} - \frac{\hat{s}\hat{t}}{\hat{u}^2}\right)$





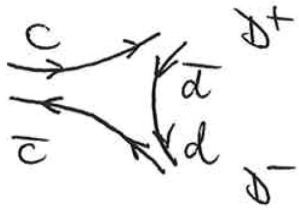


(d) quarkonium decays

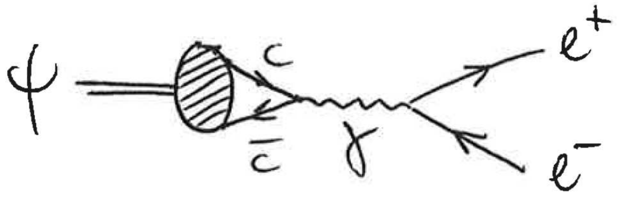
$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$  $m_\rho > 2m_\pi$ Zweig-allowed decay ⇒ large width

$\Psi = c\bar{c}$ (3097)

$\Upsilon = b\bar{b}$ (9460)

 $m_\Psi < 2m_D$ Zweig-allowed decay not possible

• leptonic decays: $\Psi \to e^+e^-, \mu^+\mu^-$ $[q\bar{q}]$



$$\Gamma(\Psi \to \ell^+\ell^-) = \frac{16\pi\alpha^2}{m_\Psi^2} Q_c^2 |\phi(0)|^2 \qquad [\leftarrow \text{positronium}]$$

+2/3 ↑ ↑ wave func. @ origin [NR]

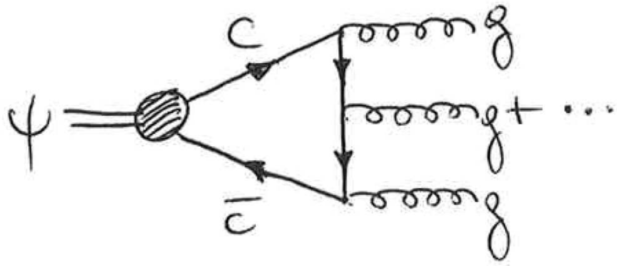
• hadronic decays: quarkonia for which Zweig-allowed decays are impossible decay into gluons → jets at high energies $[\Upsilon, \ldots]$

$1^{--} \not\to gg$: lowest ortho-channel

[Yang]

$\Psi = c\bar{c} \to 3g$

$\Upsilon = b\bar{b} \to 3g$



annihilation distance: $d \sim m_Q^{-1} \ll 1$ fm ⇒ asympt. freedom

Techn.: $\Gamma(\Psi \to ggg) = \sigma(c\bar{c} \to ggg) \times [v_R|\phi_s(0)|^2] \times \left[\frac{4}{3}\right]$

spin-average correction ↑

$\Rightarrow$ <u>width:</u> $\Gamma(Q\bar{Q} \to ggg) = \frac{160}{81}(\pi^2 - 9)\frac{\alpha_s^3(M^2)}{M^2}|\phi_s(0)|^2$

$\Psi = (0.05 \pm 0.01)$ MeV

$\Upsilon = 0.04$ MeV

quarkonia are very narrow resonances [← asympt. freedom]

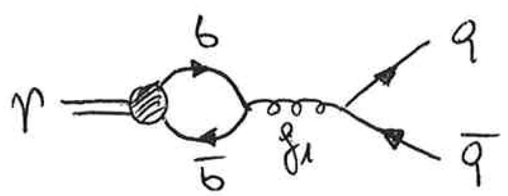
<u>Dalitz:</u> $\frac{1}{\Gamma}\frac{d\Gamma}{dx_1 dx_2} = \frac{6}{\pi^2 - 9}\,\frac{x_1^2(1-x_1)^2 + x_2^2(1-x_2)^2 + x_3^2(1-x_3)^2}{x_1^2 x_2^2 x_3^2}$

<u>jet energy:</u> $\frac{1}{\Gamma}\frac{d\Gamma}{dx_1} \approx 2x_1$

in photon channel $\Upsilon \to \gamma + gg$:

partial width $\Gamma_\gamma/\Gamma_{tot} \sim \frac{\alpha\alpha_s^2}{\alpha_s^3} \sim \frac{\alpha}{\alpha_s}$ meas. poss. small Λ

<u>color charge of gluons:</u>



if g were a $U(1)$ gauge field the dominant decay mode would be $\Upsilon \to g_{virt} \to q\bar{q}$

$\Rightarrow$ 2 jet final states = off

not observed $\Rightarrow$ $\boxed{\mathcal{C}[g] \neq 0}$

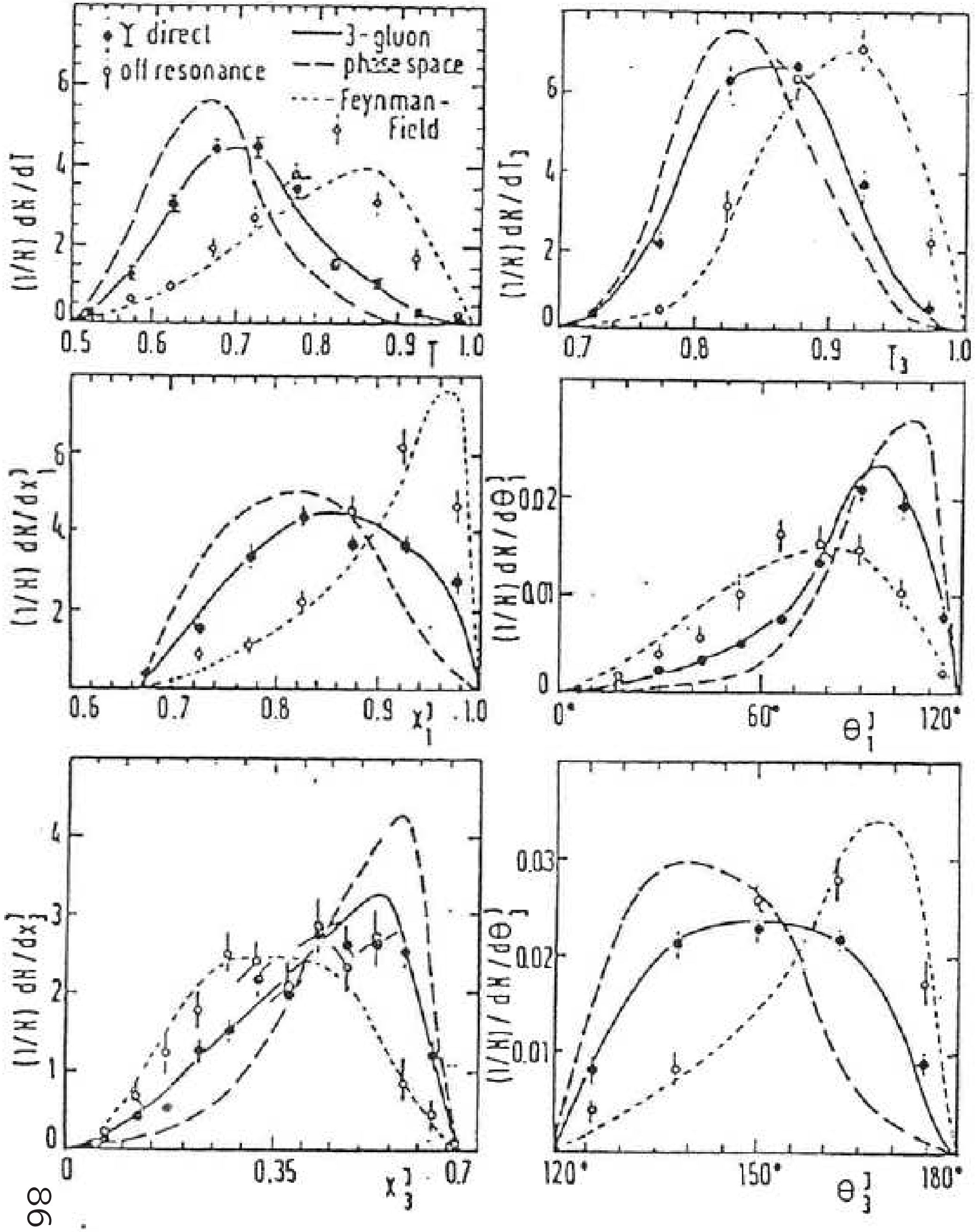
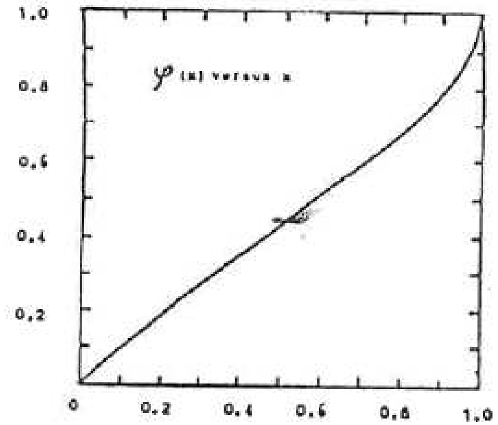
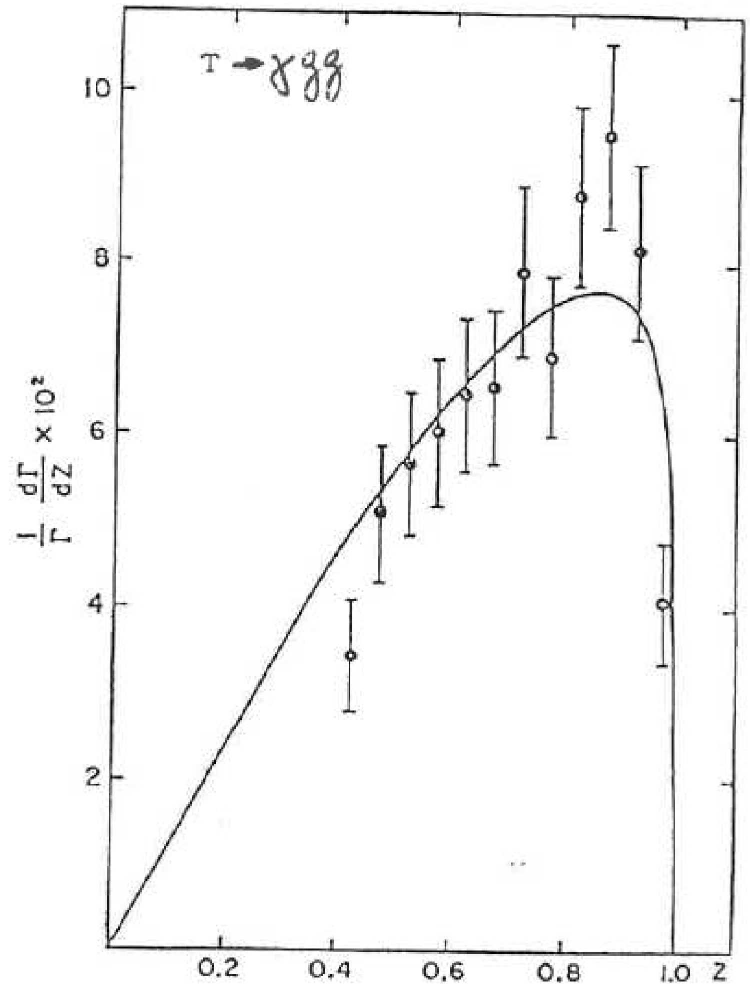


Fig. 2 Experimental distributions of thrust T, triplicity T_3, reconstructed gluon energies x_1^J, x_3^J and reconstructed angles θ_1^J, θ_3^J between gluons compared to Monte-Carlo calculations based on various models.



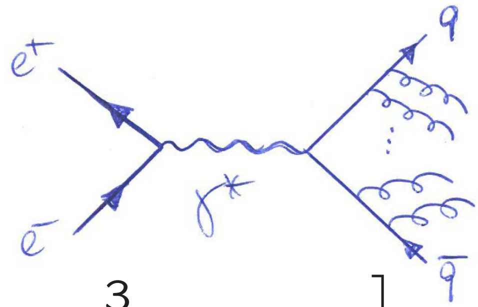


§8. Soft Gluon Resummation

soft gluon radiation:

thrust: $$\frac{1}{\sigma}\frac{d\sigma}{dT} \underset{T\to 1}{\longrightarrow} \frac{2}{3}\frac{\alpha_s}{\pi}\left\{-\frac{4}{1-T}\log(1-T)-\frac{3}{1-T}\right\}$$

singular for $T \to 1 \Rightarrow$ multi-gluon radiation



$$1 = \int_{T_{min}}^{1} \frac{1}{\sigma}\frac{d\sigma}{dT} = \int_{T_{min}}^{T} \frac{1}{\sigma}\frac{d\sigma}{dT} + f(T)$$

$$\Rightarrow f(T) = 1 - \int_{T_{min}}^{T} \frac{1}{\sigma}\frac{d\sigma}{dT} = 1 - \frac{4}{3}\frac{\alpha_s}{\pi}\left[\log^2(1-T) + \frac{3}{2}\log(1-T)\right]$$

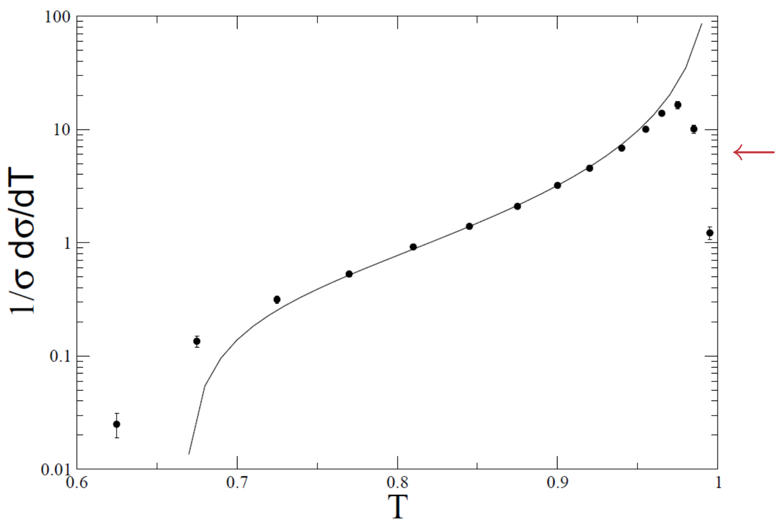
$$+\frac{8}{9}\left(\frac{\alpha_s}{\pi}\right)^2\left[\log^4(1-T) + 3\log^3(1-T)\right] + \mathcal{O}(\alpha_s^3)$$

$$\to \exp\left\{-\frac{4}{3}\frac{\alpha_s}{\pi}\left[\log^2(1-T) + \frac{3}{2}\log(1-T)\right]\right\} \underset{T\to 1}{\longrightarrow} 0$$

$$\frac{1}{\sigma}\frac{d\sigma}{dT} \underset{T\to 1}{\longrightarrow} -\frac{df(T)}{dT} = \frac{2}{3}\frac{\alpha_s}{\pi}\left\{-\frac{4}{1-T}\log(1-T) - \frac{3}{1-T}\right\} f(T)$$

$$\frac{df(T)}{dT} = W(T) f(T) \qquad\qquad \frac{1}{\sigma}\frac{d\sigma}{dT} \underset{T\to 1}{\longrightarrow} -\frac{df(T)}{dT}$$

$$W(T) = \frac{2}{3}\frac{\alpha_s}{\pi}\left\{\frac{4}{1-T}\log(1-T) + \frac{3}{1-T}\right\} + \mathcal{O}(\alpha_s^2)$$



<u>matching:</u> $[L = \log y \quad \text{with } y = 1 - T]$

$$R(y) = \int_0^y \frac{1}{\sigma}\frac{d\sigma}{dy} = 1 + \frac{\alpha_s}{\pi}\left[g_{12}L^2 + g_{11}L + g_{10} + c_1(y)\right] = f(1-T)$$

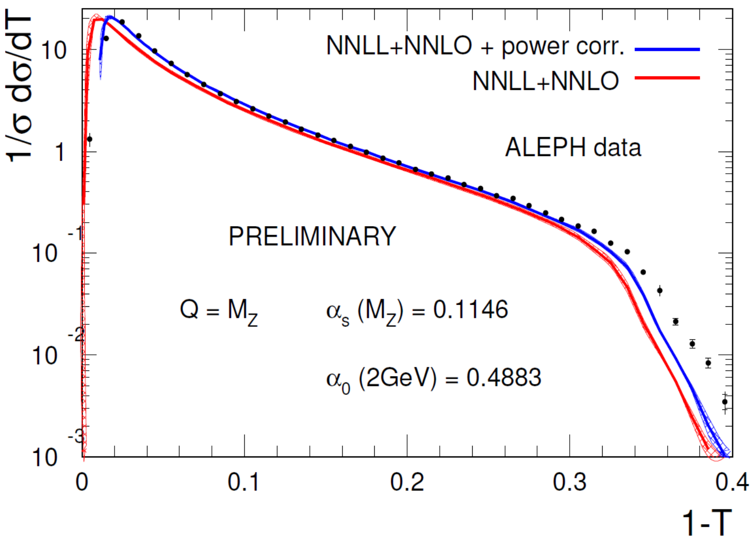
$$\text{with } c_1(y) \underset{y\to 0}{\longrightarrow} 0$$

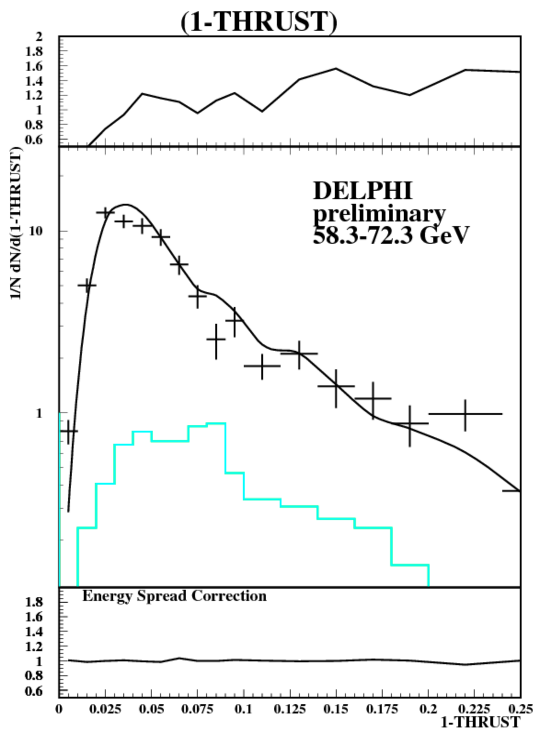
$$R(y) \to \left(1 + g_{10}\frac{\alpha_s}{\pi}\right)\exp\left\{\frac{\alpha_s}{\pi}\left[g_{12}L^2 + g_{11}L\right]\right\} + c_1(y)\frac{\alpha_s}{\pi} \qquad [\text{“}R\text{–matching”}]$$

$$R(y) \to \exp\{R(y) - 1\} \qquad [\text{“}\log R\text{–matching”}]$$

here: $\quad g_{12} = -\frac{4}{3} \qquad g_{11} = -2$

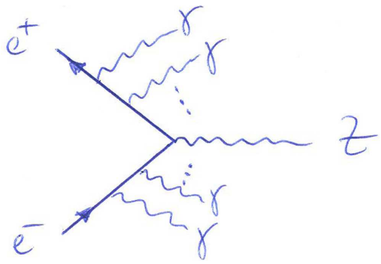
$$\frac{1}{\sigma}\frac{d\sigma}{dy} = \frac{dR(y)}{dy}$$





$$\langle 1 - T \rangle = 0.0685 \pm 0.0015 \pm 0.0012$$

• <u>QED: $e^+e^- \to Z$</u>



$$\sigma = \int_{z_0}^{1} dz\; G(z)\sigma(zs) \qquad z_0 = \frac{M_Z^2}{s} \qquad L = \log\frac{s}{m_e^2}$$

$$G(z) \sim \delta(1-z) + \frac{\alpha}{\pi}\left[a_{11}L + a_{10}\right] + \left(\frac{\alpha}{\pi}\right)^2\left[a_{22}L^2 + a_{21}L + a_{20}\right] + \cdots$$

$$a_{11} = \frac{3}{2}\delta(1-z) + \frac{2}{(1-z)_+} \qquad a_{10} = 2[\zeta(2)-1]\delta(1-z) - \frac{2}{(1-z)_+}$$

$$R(y) = \int_0^y dy\,\frac{1}{\sigma}\,\frac{d\sigma}{dy}\bigg|_{y=1-z} = 1 + \frac{\alpha}{\pi}\left\{2(L-1)\log(1-z) + \frac{3}{2}L + 2\zeta(2) - 2\right\}$$

$$\to e^{\beta\log(1-z)}\left[1 + \frac{\alpha}{\pi}\left(\frac{3}{2}L + 2\zeta(2) - 2\right)\right] + \cdots$$

$$G(z) = \frac{dR(y)}{dy} = W(y)R(y) + \cdots\bigg|_{y=1-z} \qquad \beta = 2\frac{\alpha}{\pi}(L-1)$$

$$= \beta(1-z)^{\beta-1}\left[1 + \frac{\alpha}{\pi}\left(\frac{3}{2}L + 2\zeta(2) - 2\right)\right] - \frac{1+z}{2}\beta + \mathcal{O}(\alpha^2)$$

⇒ sing. at $z = 1$ regularized by multi-soft-photon radiation

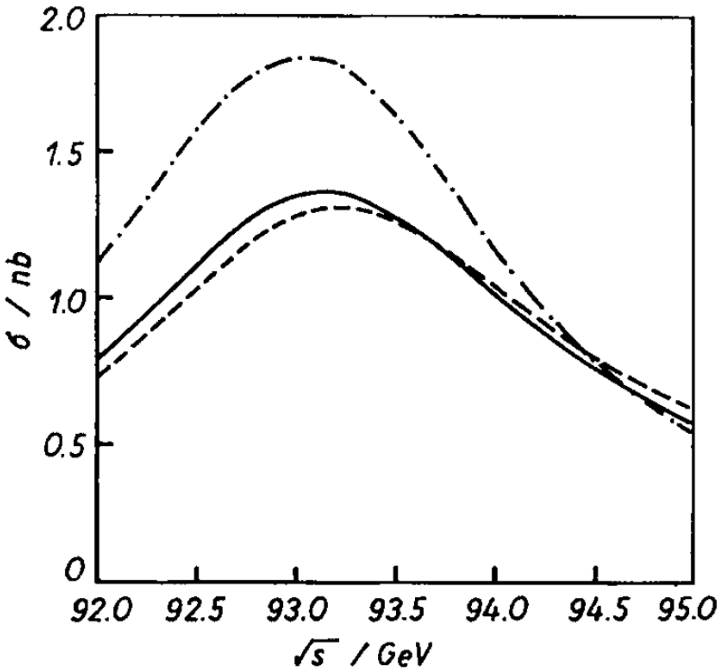


Fig. 6.5. Total cross section for $e^+e^- \to \mu^+\mu^-$
—·—·— Born approximation;
- - - - - - $O(\alpha)$ corrected
———— $O(\alpha^2)$ corrected

• QCD: Drell–Yan

$$\sigma(\tau_0) = \sum_{i,j} \int_{\tau_0}^{1} d\tau \;\; (f_i \otimes f_j)\, \hat{\sigma}_{ij}$$

with $f \otimes g = \int_x^1 \frac{dz}{z}\; f\left(\frac{x}{z}\right) g(z)$

$$\hat{\sigma}_{ij} = \sigma_0 \;\; \rho_{ij}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)$$

$$\rho_{q\bar{q}}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) \sim \delta(1-z) + C_F \frac{\alpha_s}{\pi} \left\{ \left[\frac{3}{2} \log \frac{Q^2}{\mu^2} + 2\zeta(2) - 4\right] \delta(1-z) \right.$$

$$\left. +2 \left(\frac{\log \frac{Q^2(1-z)^2}{\mu^2}}{1-z} \right)_+ + \cdots \right\}$$

$z \to 1$: soft region $\to$ Sudakov evolution equation:

$$Q^2 \frac{d\rho_{q\bar{q}}}{dQ^2} = W \otimes \rho_{q\bar{q}}$$

• Mellin-moments: $\tilde{\sigma}(N) = \int_0^1 d\tau_0 \;\; \tau_0^{N-1} \sigma(\tau_\mu)$

$$\tilde{\sigma}(N) = i\sigma_0 \sum_{i,j} \tilde{f}_i(N+1) \tilde{f}_j(N+1) \tilde{\rho}_{ij}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)$$

$$\Rightarrow Q^2 \frac{d\tilde{\rho}_{q\bar{q}}}{dQ^2} = \tilde{W} \;\; \tilde{\rho}_{q\bar{q}}$$

$$\left(\frac{\log^n(1-z)}{1-z}\right)_+ \to \frac{(-1)^{n+1}}{n+1} \log^{n+1} \tilde{N} \qquad (\tilde{N} = N e^{\gamma_E})$$

$$\Rightarrow \boxed{z \to 1 \leftrightarrow N \to \infty}$$

$$\tilde{\rho}_{q\bar{q}} = \exp\left\{ \int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \tilde{W}\left(N, \frac{q^2}{\mu^2}, \alpha_s(\mu^2)\right) \right\}$$

$$\tilde{\rho}_{q\bar{q}} = 1 + C_F \frac{\alpha_s}{\pi} \left\{ \frac{3}{2} \log \frac{Q^2}{\mu^2} + 2\zeta(2) - 4 + 2 \log \tilde{N} \log \frac{\tilde{N}\mu^2}{Q^2} + \mathcal{O}\left(\frac{1}{N}\right) \right\}$$

• general result after resummation and mass factorization:

$$\tilde{\rho}_{ij}\left(N,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right) = C\left(\frac{Q^2}{\mu^2},\alpha_s(Q^2)\right)\times$$
$$\exp\left\{\int_0^1 dz\frac{z^{N-1}-1}{1-z}\left[2\int_{\mu^2}^{(1-z)^2Q^2}\frac{dq^2}{q^2}A[\alpha_s(q^2)]+D[\alpha_s((1-z)^2Q^2)]\right]\right\}$$

$$C\left(\frac{Q^2}{\mu^2},\alpha_s(Q^2)\right) = 1+C_F\frac{\alpha_s}{\pi}\left[\frac{3}{2}\log\frac{Q^2}{\mu^2}+4\zeta(2)-4+2\gamma_E\right]+\mathcal{O}(\alpha_s^2)$$

$$A(\alpha_s)=\sum_{n=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n A^{(n)} \qquad D(\alpha_s)=\sum_{n=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n D^{(n)}$$

$$A^{(1)}=C_F \qquad A^{(2)}=\frac{C_F}{2}\left[C_A\left(\frac{67}{18}-\zeta(2)\right)-\frac{5}{9}N_F\right]$$

$$D^{(1)}=0 \qquad D^{(2)}=\frac{C_FC_A}{16}\left(-\frac{1616}{27}+56\,\zeta(3)+\frac{176}{3}\zeta(2)\right)$$
$$+\frac{C_FN_F}{16}\left(\frac{224}{27}-\frac{32}{3}\zeta(2)\right)$$

• moments after integration:

$$G_{DY}^N=\log\frac{\tilde{\rho}_{q\bar{q}}}{C}=Lg_1(\lambda)+g_2(\lambda)+\frac{\alpha_s}{\pi}g_3(\lambda)+\cdots$$

$$L=\log N \quad \lambda=b_0L\frac{\alpha_s}{\pi} \quad b_0=\frac{33-2N_F}{12} \quad b_1=\frac{153-19N_F}{24}$$

$$g_1(\lambda)=\frac{A^{(1)}}{b_0\lambda}[2\lambda+(1-2\lambda)\log(1-2\lambda)]$$

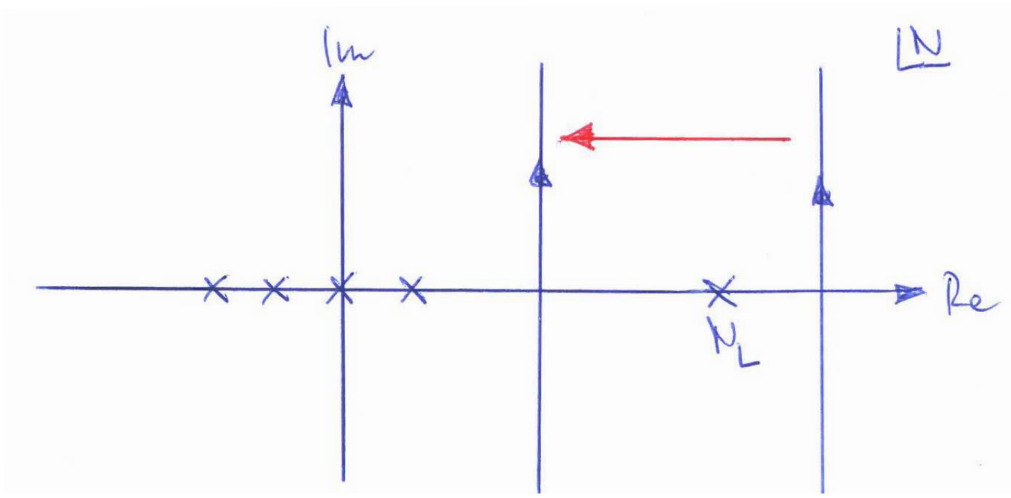
$$g_2(\lambda)=-2\frac{A^{(1)}\gamma_E}{b_0}\log(1-2\lambda)+\frac{A^{(1)}b_1}{b_0^3}\left[2\lambda+\log(1-2\lambda)\right.$$
$$\left.+\frac{1}{2}\log^2(1-2\lambda)\right]-\frac{A^{(2)}}{b_0^2}[2\lambda+\log(1-2\lambda)]+\frac{A^{(1)}}{b_0}\log(1-2\lambda)\log\frac{Q^2}{\mu^2}$$

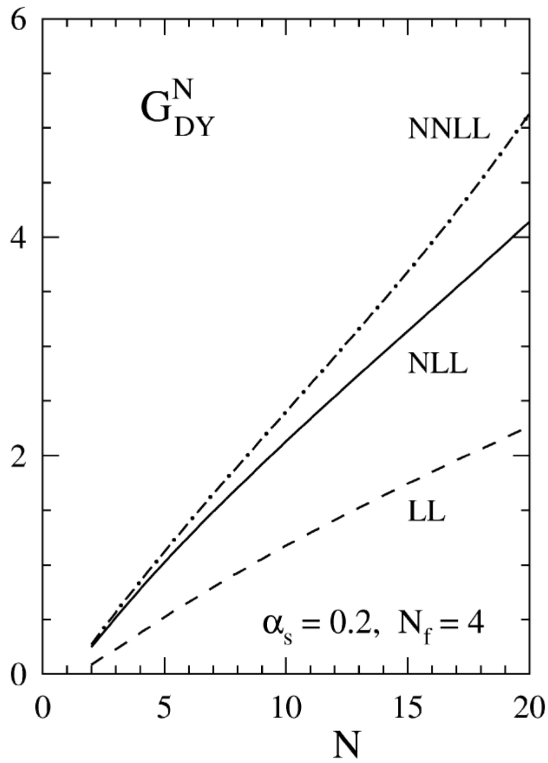
• Mellin inversion:

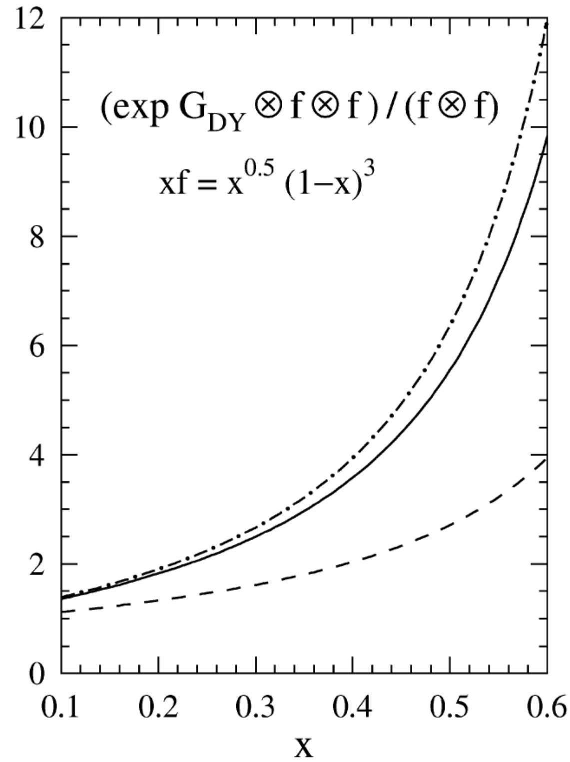
$$\rho_{q\bar{q}}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dN\, z^{-N}\, \tilde{\rho}_{q\bar{q}}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)$$

singularity at $N = N_L = \exp\left\{\frac{\pi}{2b_0\alpha_s(\mu^2)}\right\}$ [← Landau pole]

minimal presciption:







C. QCD AT LARGE DISTANCES

§1. Confinement Potential

non-perturbative region: integration via path integrals

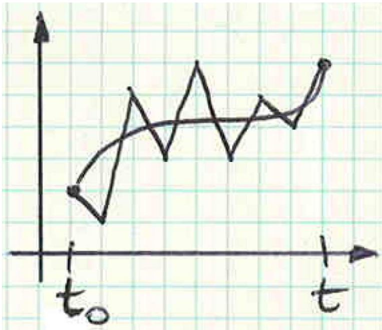
QM: $\langle x,t|x_0,t_0\rangle = \langle x|e^{-iH(t-t_0)}|x_0\rangle$

$$= \int \prod_i dx_i \langle x|e^{-iH\epsilon}|x_n\rangle\langle x_n|\cdots|x_1\rangle\langle x_1|e^{-iH\epsilon}|x_0\rangle$$

FT: $\langle x_2|e^{-iH\epsilon}|x_1\rangle = \langle x_2|e^{-i\frac{\mathbb{P}^2}{2m}\epsilon}|x_1\rangle$

$$\sim \int dp\ \langle x_2|e^{-i\frac{p^2}{2m}\epsilon}|p\rangle\langle p|x_1\rangle$$

$$\sim \int dp\ e^{-i\frac{p^2}{2m}\epsilon + ip(x_2-x_1)}$$

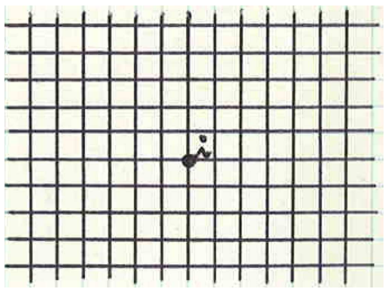


$$\sim \exp\left\{i\frac{m}{2}\epsilon\left(\frac{x_2-x_1}{\epsilon}\right)^2\right\} \sim \exp\left\{i\frac{m}{2}\epsilon\dot{x}^2\right\} \sim \exp\{i\epsilon\mathcal{L}\}$$

$\langle x,t|x_0,t_0\rangle = \int \mathcal{D}x\ e^{iS}$ p'amplitude = sum over all histories with weight e^{iS}

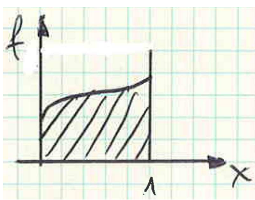
QFT: $\langle 0|T\{\phi(x_2)\phi(x_1)\}|0\rangle \sim \int \prod d\phi_i\ \phi_2\phi_1\ e^{iS(\phi)}$

solution of integrals via Monte-Carlo methods:
[after Euklidisation $t \to -ix_4$: $e^{iS} \to e^{-S}$]



1.) simple integral:

$$\int_0^1 dx\ f(x) = ?$$

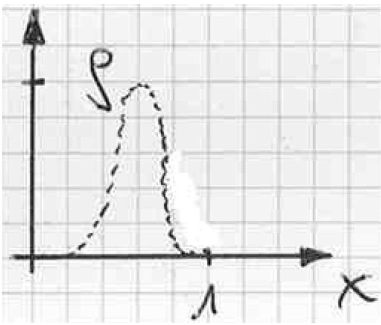


(i) choose x_i randomly distributed; N points
(ii) calculate $f(x_i) = f_i$

$$\to \int_0^1 dx\ f(x) = \frac{1}{N}\sum_i f_i$$

2.) importance sampling:

$\int_0^1 dx\ \rho(x) f(x)$ ρ significant only in small part of phase space



$$I = \int_0^1 dx \left[\int_0^1 dy\ \Theta(\rho(x) - y) \right] f(x)$$

(i) choose x_i randomly; calculate ρ_i

(ii) choose y_i randomly; <u>if</u> $\rho_i > y_i$ then calculate f_i, $i \in \mathbb{N}_+$

$$\to I = \frac{1}{N} \sum_{i \in \mathbb{N}_+} f_i$$

3.) high dimension, weight e^{-S_E}, simple observable:
generate sequence of configurations $\{\phi\}$ with distribution e^{-S_E}; measure observable $\mathcal{O}$ over configurations:
$\langle \mathcal{O} \rangle = \sum_i \mathcal{O}_i / \sum_i \mathbb{1}$

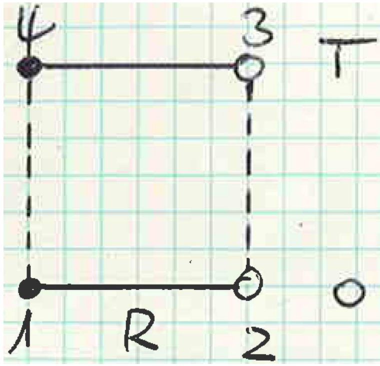
<u>Theorem:</u> Starting from an arbitrary configuration and modifying it in consecutive steps with conditional probability $P(\mathcal{C}', \mathcal{C})$ such that equilibrium $\to$ equilibrium, $e^{-S(\mathcal{C}')} = \sum_{\mathcal{C}} P(\mathcal{C}', \mathcal{C}) e^{-S(\mathcal{C})}$, then the system tends to equilibrium automatically.

<u>Metropolis:</u> $P(\mathcal{C}', \mathcal{C}) = \begin{cases} 1 & \text{for } S' \le S \\ e^{-(S'-S)} & \text{for } S' > S \end{cases}$

WILSON Loop

barbell operator: $h = \bar{Q}(y) e^{ig_s \int_x^y ds_\mu G^\mu} Q(x)$

[gauge invariant]



$\langle 0 | h^*(T) h(0) | 0 \rangle$

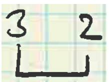
$= \sum_n \langle 0 | e^{HT} h^*(0) e^{-HT} | n \rangle \langle n | h(0) | 0 \rangle$

$= \sum_n |\langle 0 | h^*(0) | n \rangle|^2 e^{-E_n T}$

$\sim |\langle 0 | h(0) | (\bar{Q}Q)_0 \rangle|^2 e^{-E_{min}(R) T}$ $\qquad E_{min}(R) = 2m + V(R)$

$= \langle 0 | \bar{Q}_4 \mathcal{P}_{43} Q_3 \cdot \bar{Q}_2 \mathcal{P}_{21} Q_1 | 0 \rangle$ $\qquad$ [Q_i = external sources]

$= \int \mathcal{D}G \; {}_Q\langle 0 | \bar{Q}_4 \mathcal{P}_{43} Q_3 \cdot \bar{Q}_2 \mathcal{P}_{21} Q_1 | 0 \rangle_Q \; e^{-S_G} + \mathcal{O}\left(\frac{1}{m}\right)$

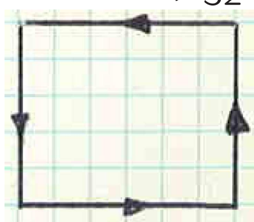


= quark propagator in background field G

$\left[\gamma^0 (i\partial_0 - g_s G_0) - m\right] S(x, y; G) = \delta_4(x - y)$

$\Rightarrow iS(x, y; G) = \Theta(x^0 - y^0) \delta_3(\vec{x} - \vec{y}) e^{-im(x^0 - y^0)} e^{-ig_s \int_y^x ds_\mu G^\mu} \propto \mathcal{P}_{32}$

$\longrightarrow$ $\boxed{e^{-V(R)T} \sim \int \mathcal{D}G \prod \mathcal{P} \; e^{-S}}$ integral solved with MC methods



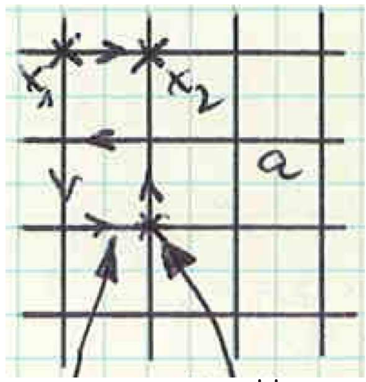
EVALUATION of $SU(3)$ without quark loops:

(i) matter fields defined on lattice points

(ii) gauge fields defined on links

Schwinger current $j = \bar{\psi}(x_2) e^{ie \int_{x_1}^{x_2} dy_\mu G^\mu} \psi(x_1)$

[non-local but gauge invariant]



gauge fields $\qquad$ matter

$G_\mu \leftrightarrow U_\mu = e^{ieaG_\mu} \to e^{iaA_\mu}$ link variable

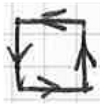
compact version suggested by Bohm–Aharonov effect

$[e^{i\oint dy_\mu A^\mu}$ = relevant variable]

Action:

$$S = -\frac{1}{2}\int d^4x\, TrG^2_{\mu\nu} = -\frac{1}{2}\int d^4x\, Tr\{\partial_\nu G_\mu - \partial_\mu G_\nu - ig_s[G_\mu, G_\nu]\}^2$$

$$\underset{g_sG=\widehat{G}\to G}{=} -\frac{1}{2g_s^2}\int d^4x\; Tr\{\partial_\nu G_\mu - \partial_\mu G_\nu - i[G_\mu, G_\nu]\}^2$$

lattice: plaquette variable: $U_\square = \prod U$ 

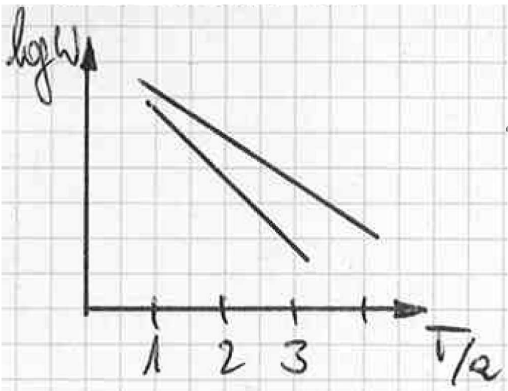
$$S = \frac{6}{g_s^2}\left\{1 - \frac{1}{6}\sum_{\mathcal{P}} Tr[U_\square + U_\square^\dagger]\right\}$$

expansion up to 2nd order

→ continuum action

S lattice-gauge invariant

MEASUREMENT:



slope: V in lattice units

Fit ansatz:

physical units: $V(R) = -\frac{\widehat{\alpha}_s}{R} + \sigma R$

σ = string tension

lattice units: $aV = -\frac{\widehat{\alpha}_s}{R/a} + \sigma a^2 \frac{R}{a}$

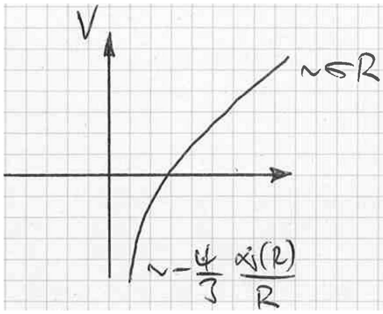
Fit: σa^2 measured on lattice $\underset{a\in am_\rho}{\longrightarrow}$ $\sqrt{\sigma} \approx 400$ MeV determined

$\widehat{\alpha}_s \approx 0.25$ measured

measurement res. in qualitative agreement with expectation

String tension:

(a) Richardson-potential:



$$\left.\begin{array}{l}\vec{q}^2 \text{ large}\\ R \text{ small}\end{array}\right\}\quad \begin{array}{l} V(\vec{q}^2) \approx -\dfrac{4}{3}\dfrac{\alpha_s(\vec{q}^2)}{\vec{q}^2} \\[2ex] V(R) \approx -\dfrac{4}{3}\dfrac{\alpha_s(R)}{R}\end{array} \qquad \begin{array}{l}\alpha_s(\vec{q}^2) = \dfrac{12\pi}{(33-2N_F)\log\frac{\vec{q}^2}{\Lambda^2}} \\[2ex] \alpha_s(R) = \dfrac{12\pi}{(33-2N_F)\log\frac{1}{R^2\Lambda^2}}\end{array}$$

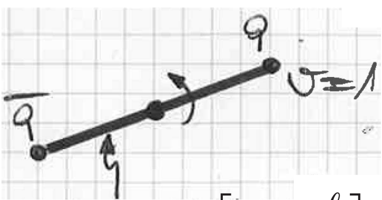
$$\left.\begin{array}{l}\vec{q}^2 \text{ small}\\ R \text{ large}\end{array}\right\}\quad \begin{array}{l} V(\vec{q}^2) \approx -\dfrac{8\pi\sigma}{\vec{q}^4} \\[2ex] V(R) \approx \sigma R\end{array}$$

interpolation: Richardson-potential

$$V(R) = -\frac{4}{3}\,\frac{48\pi^2}{33-2N_F}\int\frac{d^3\vec{q}}{(2\pi)^3}\,\frac{e^{i\vec{q}\vec{r}}}{\vec{q}^2\log\left[1+\frac{\vec{q}^2}{\Lambda^2}\right]}$$

$\longrightarrow \sigma \approx \Lambda^2 \approx (400\text{ MeV})^2$: quarkonium spectroscopy

(b) Meson string rotator:



gluon tube $\left[\dfrac{v}{c} = \dfrac{\ell}{L}\right]$

$$dm = \underset{\substack{\uparrow\\ \text{energy density}}}{\gamma\sigma d\ell} = \frac{\sigma d\ell}{\sqrt{1-\frac{v^2}{c^2}}} = L\frac{\sigma d\frac{\ell}{L}}{\sqrt{1-\left(\frac{\ell}{L}\right)^2}} = L\frac{\sigma dx}{\sqrt{1-x^2}}$$

mass, energy

$$dj = \ell dp = \ell v dm = L^2\frac{\sigma x^2 dx}{\sqrt{1-x^2}}$$

angular mom.

$$\left.\begin{array}{l} m = \pi L\sigma \\ j = \dfrac{\pi}{2}L^2\sigma\end{array}\right\}\ \boxed{j = \frac{m^2}{2\pi\sigma}}$$

Chew–Frautschi plot: linear spin-mass^2 relation

$\sigma = (420\text{ MeV})^2$

restauration of rotational invariance

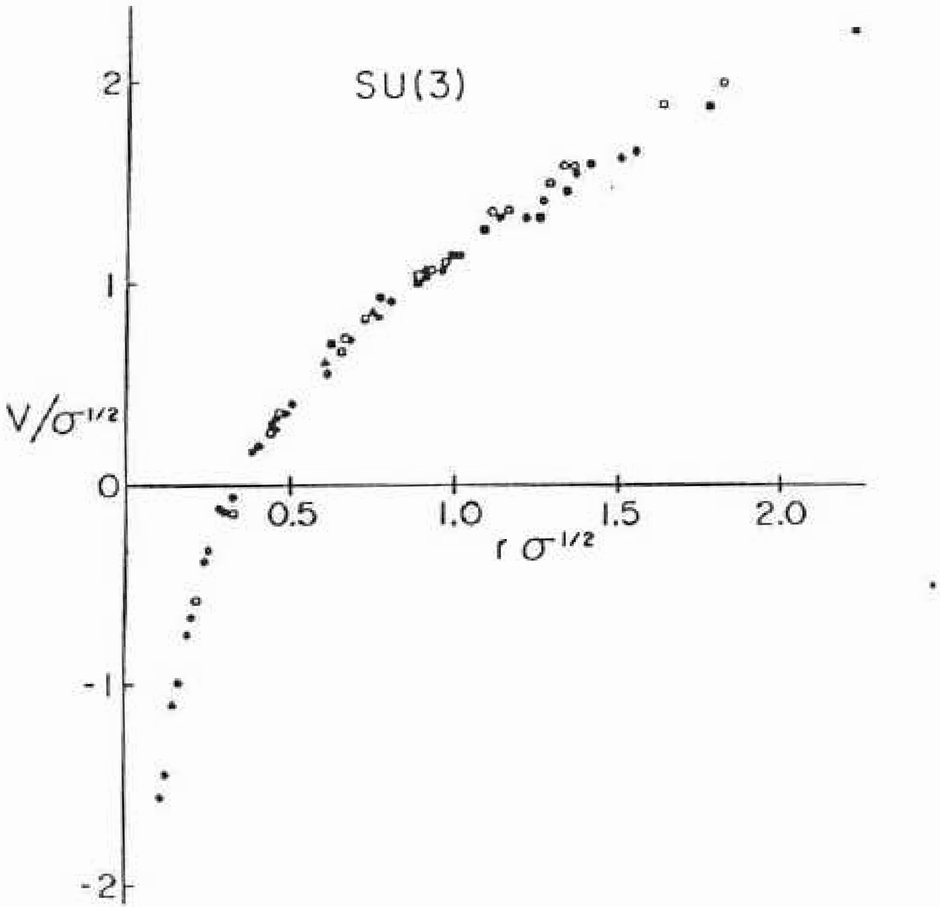


Figure 2: The q-q̄ potential as obtained from different numerical simulations.

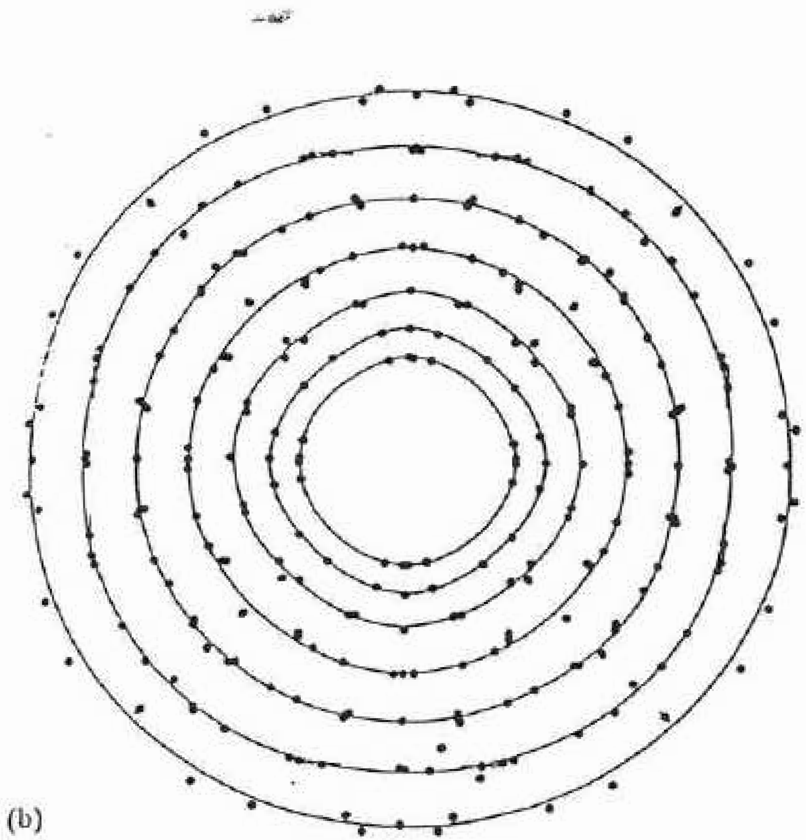


Fig. 1. Restoration of rotational invariance from (a) $\beta = 2, n_s = 8, n_t = 4$ to (b) $\beta = 2.25, n_s = 16, n_t = 6$; the equipotential curves are obtained through fits as described in the text.

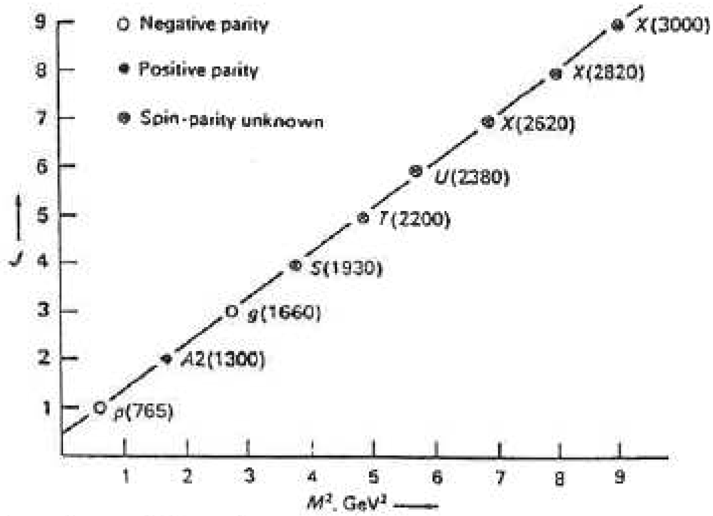


Fig. 7.34 Chew-Frautschi plot of nonstrange meson resonances, of $I = 1$ and spin, parity and G-parity, $J^{PG} = 1^{-+}, 2^{+-}, 3^{-+} \cdots$. The quantum numbers of only the first three states are known at present, the remainder having been plotted at the nearest integer spin value. The masses of the S, T, U, and X bosons are taken from Fig. 7.22.

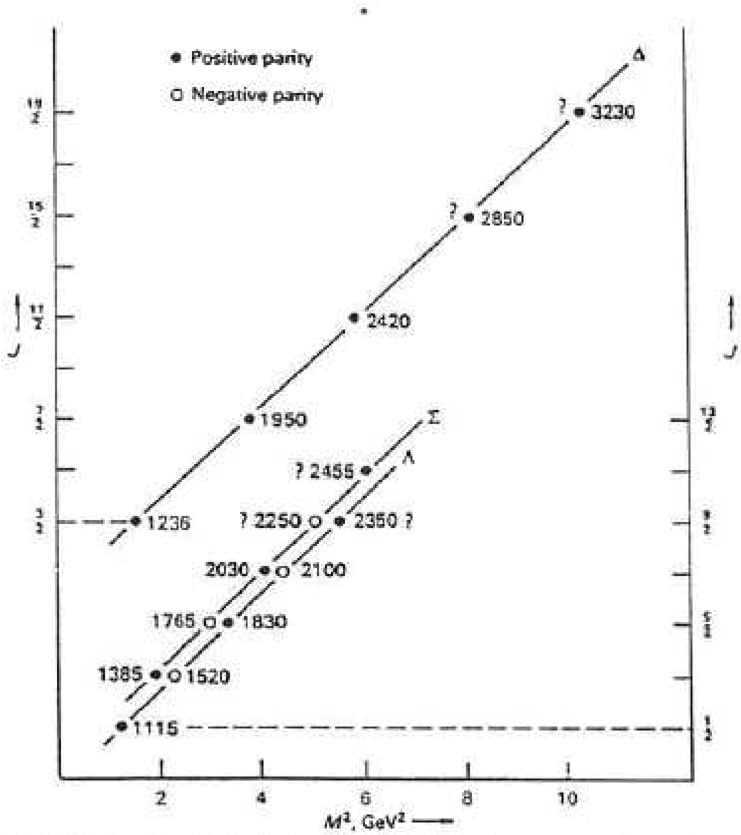


Fig. 7.33 Chew-Frautschi plots of fermion Regge trajectories. The trajectory marked Δ consists of the sequence $I = \frac{3}{2}$, $S = 0$, and $J^P = \frac{3}{2}^+, \frac{7}{2}^+, \frac{11}{2}^+ \cdots$; that marked Λ of the sequence $I = 0$, $S = -1$, $J^P = \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+ \cdots$; and that marked Σ of the sequence $I = 1$, $S = -1$, $J^P = \frac{3}{2}^+, \frac{5}{2}^-, \frac{7}{2}^+ \cdots$; resonances for which the spin-parity is not firmly established are indicated by a question mark.

§2. Chiral Invariance

operator transformations: $\phi_i' = e^{i\vec{\alpha}\vec{Q}}\ \phi\ e^{-i\vec{\alpha}\vec{Q}} = e^{i\vec{\alpha}\vec{T}}\ \phi$

flavor operator ↑ ↑ SU_N gener.

Noether current: $j_\mu^a = -\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_i)}\frac{\delta \phi_i}{\delta \alpha^a} - \frac{\partial \mathcal{L}}{\partial(\partial^\mu \bar{\phi}_i)}\frac{\delta \bar{\phi}_i}{\delta \alpha^a}$

$$\Rightarrow \boxed{\partial^\mu j_\mu^a = -\frac{\delta \mathcal{L}}{\delta \alpha^a}} \quad \boxed{Q^a = \int d^3\vec{x} j_0^a(x)}$$

Lagrangian invariant: $\frac{\delta \mathcal{L}}{\delta \alpha^a} = 0 \Rightarrow \partial^\mu j_\mu^a = 0 \Rightarrow \dot{Q}^a = 0$

quantum flavordynamics: $q = \begin{pmatrix} u \\ d \\ -- \\ s \end{pmatrix}$

mass matrix: $M_{ij} = m_i \delta_{ij}$

$$\mathcal{L} = \sum_j \bar{q}_j (i\partial\!\!\!/ - m_j) q_j = \bar{q}(i\partial\!\!\!/ - M) q$$

(i) vector current: $q' = e^{i\vec{\alpha}\vec{T}} q \Rightarrow \bar{q}' = \bar{q}\ e^{-i\vec{\alpha}\vec{T}}$

$$\mathcal{L}' = \bar{q}\left[i\partial\!\!\!/ - e^{-i\vec{\alpha}\vec{T}} M e^{i\vec{\alpha}\vec{T}}\right] q \neq \mathcal{L}$$

$$\boxed{j_\mu^a = \bar{q}\gamma_\mu T^a q} \quad \boxed{Q^a = \int d^3\vec{x}\ q^\dagger T^a q}$$

divergence: $\partial^\mu j_\mu^a = -\frac{\delta \mathcal{L}}{\delta \alpha^a} = \bar{q}\,[M, iT^a]\,q = (m_j - m_k)\bar{q}_j i T_{jk}^a q_k$

(ii) axial vector current: $q' = e^{i\vec{\alpha}\vec{T}\gamma_5} q \Rightarrow \bar{q}' = \bar{q}\ e^{i\vec{\alpha}\vec{T}\gamma_5}$

$$\mathcal{L}' = \bar{q}\left[i\partial\!\!\!/ - e^{i\vec{\alpha}\vec{T}\gamma_5} M e^{i\vec{\alpha}\vec{T}\gamma_5}\right] q \neq \mathcal{L}$$

$$\boxed{j_{5\mu}^a = \bar{q}\gamma_\mu\gamma_5 T^a q} \quad \boxed{Q_5^a = \int d^3\vec{x}\ q^\dagger \gamma_5 T^a q}$$

divergence: $\partial^\mu j_{5\mu}^a = \bar{q}\gamma_5 \{M, iT^a\}\, q = (m_j + m_k)\bar{q}_j \gamma_5 i T_{jk}^a q_k$

• charge/current algebra for equal times $[SU_N \times SU_N]$:

$$\boxed{\begin{aligned} [Q^a(t), Q^b(t)] &= i f_{abc} Q^c(t) \\ [Q_5^a(t), Q_5^b(t)] &= i f_{abc} Q^c(t) \\ [Q^a(t), Q_5^b(t)] &= i f_{abc} Q_5^c(t) \end{aligned}}$$

$$Q_\pm^a = \frac{Q^a \pm Q_5^a}{2} = \int d^3\vec{x} q^\dagger \frac{1 \pm \gamma_5}{2} T^a q$$

$$\boxed{\begin{aligned} [Q_\pm^a(t), Q_\pm^b(t)] &= i f_{abc} Q_\pm^c(t) \\ [Q_+^a(t), Q_-^b(t)] &= 0 \end{aligned}}$$

independent of invariance status the charges Q^a, Q_5^a fulfill the Lie-Algebra above for equal times due to the kanonical commutation rules.

CR: $\{q_{i\alpha}^\dagger, q_{i\beta}\}_{ET} = \delta_{\alpha\beta}\delta_3(\vec{x} - \vec{y})$

$$[T^a, T^b] = i f_{abc} T^c$$

• current conservation: (i) $m_j = m_k \Rightarrow \partial^\mu j_\mu^a = 0$ [i.g. $\partial^\mu j_{5\mu}^a \neq 0$]

(ii) $m_j = 0 \Rightarrow \partial^\mu j_{5\mu}^a = \partial^\mu j_\mu^a = 0$ [chirality]

chiral invariance: $M \to 0 \Rightarrow Q^a, Q_5^a$ conserved

→ Heisenberg multiplets: $q|0\rangle \to e^{i\vec{\alpha}\vec{Q}}\, q\, e^{-i\vec{\alpha}\vec{Q}} \underbrace{e^{i\vec{\alpha}\vec{Q}}|0\rangle}_{=|0\rangle} = e^{i\vec{\alpha}\vec{T}}\, q|0\rangle$

$\dot{\vec{Q}}_5 = i[H, \vec{Q}_5] = \vec{0}$

parity:

$$\begin{aligned} &\mathcal{P}\vec{Q}\mathcal{P}^{-1} = \vec{Q} && \mathcal{P}\vec{Q}_5\mathcal{P}^{-1} = -\vec{Q}_5 \\ &H|z\rangle = m_z|z\rangle && HQ_5|z\rangle = m_z Q_5|z\rangle \Rightarrow \text{degeneracy} \\ &\mathcal{P}|z\rangle = |z\rangle && \mathcal{P}Q_5|z\rangle = -Q_5|z\rangle \end{aligned}$$

$\Rightarrow$ parity doublets

→ Nambu realization: chiral $SU_N \times SU_N$ spontaneously broken

$|0\rangle \to e^{i\vec{\alpha}\vec{T}\gamma_5}|0\rangle$ [vacuum not invariant]

• condensate:

(i) Heisenberg: $e^{i\vec{\alpha}\vec{Q}_5}|0\rangle = |0\rangle$

$$\langle 0|\bar{\psi}\psi|0\rangle = \langle 0|e^{-i\vec{\alpha}\vec{Q}_5}e^{i\vec{\alpha}\vec{Q}_5}\ \bar{\psi}\ e^{-i\vec{\alpha}\vec{Q}_5}e^{i\vec{\alpha}\vec{Q}_5}\ \psi\ e^{-i\vec{\alpha}\vec{Q}_5}e^{i\vec{\alpha}\vec{Q}_5}|0\rangle$$

$$= \left(e^{2i\vec{\alpha}\vec{T}\gamma_5}\right)_{ab} \langle 0|\bar{\psi}_a\psi_b|0\rangle$$

$$\Rightarrow \langle 0|\bar{\psi}\psi|0\rangle = 0$$

(ii) Nambu: $\langle 0|\bar{\psi}\psi|0\rangle = \left(e^{2i\vec{\alpha}\vec{T}\gamma_5}\right)_{ab}\ {}_{\alpha}\langle 0|\bar{\psi}_a\psi_b|0\rangle_{\alpha}$

$$\Rightarrow \langle 0|\bar{\psi}\psi|0\rangle \neq 0 \text{ possible}$$

Goldstone Theorem:

N = dimension of algebra belonging to the symmetry group of the full Lagrangian

M = dimension of algebra that leaves vacuum invariant after symmetry breaking

$\Rightarrow$ there are $N - M$ massless goldstones in the theory

Ex.: $SU_{2L} \times SU_{2R} : N = 6 \rightarrow SU_{2L+R} : M = 3$

$\Rightarrow$ 3 goldstones: $\pi^{\pm}, \pi^0$ $[u, d$ quarks$]$ $\left[\frac{m_\pi^2}{m_\rho^2} \approx 2\%\right]$

$SU_{3L} \times SU_{3R} : N = 16 \rightarrow SU_{3L+R} : M = 8$

$\Rightarrow$ 8 goldstones: $\pi^{\pm}, \pi^0, K^{\pm}, K^0, \bar{K}^0, \eta$

§3. PCAC Hypothesis

• Haag's theorem:

Let ϕ be an operator with the properties

(i) quantum numbers correct

(ii) $|\langle 0|\phi|1\rangle|^2 = 1$ [normalization]

Then ϕ can be used as an effective field operator.

• pion field: $\langle 0|j^a_{5\mu}(x)|\pi^b(p)\rangle = if_\pi p^\mu e^{-ipx}$

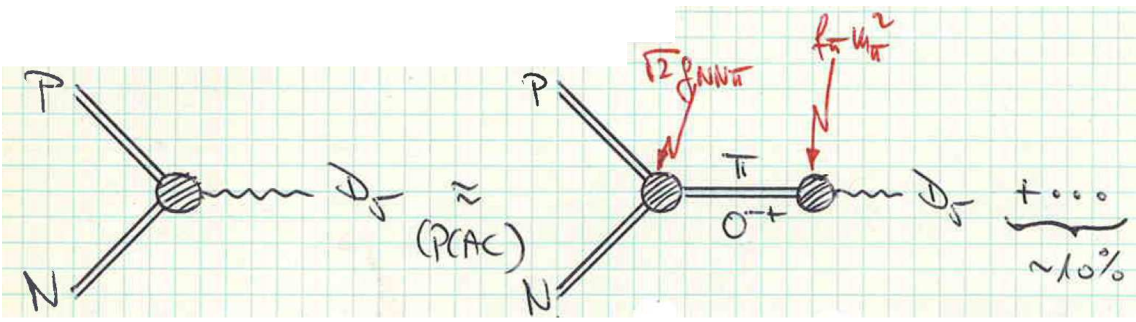
$$\Rightarrow \langle 0|D^a_5(0)|\pi^b(p)\rangle = \langle 0|\partial^\mu j^a_{5\mu}(0)|\pi^b(p)\rangle = f_\pi m^2_\pi$$

$$\Rightarrow \boxed{\phi^a_\pi(x) = \frac{D^a_5(x)}{f_\pi m^2_\pi}}$$

PCAC hypothesis:

Wherever the divergence of an axial vector current appears, it can be substituted by a 1–pion field. [Pion pole-dominance]

Ex.: (i) Nucleon form factors:

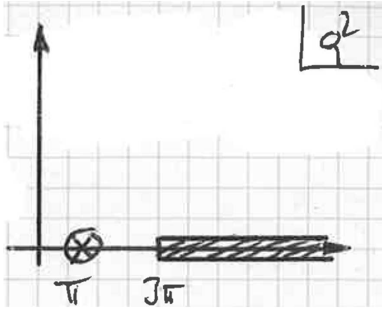


$$\langle P|j_{5\mu}(0)|N\rangle = \bar{u}_P \left[\gamma_\mu\gamma_5 g_A(q^2) + q_\mu\gamma_5 g'_A(q^2)\right] u_N$$

$$\Rightarrow \langle P|D_5(0)|N\rangle = i\bar{u}_P \left[-2m_N\ g_A(q^2) - q^2 g'_A(q^2)\right] \gamma_5\, u_N$$

$$\equiv i\bar{u}_P\ D(q^2)\gamma_5\, u_N$$

$$D(q^2) = \frac{1}{\pi} \int_{cut} dq'^2 \ \frac{\Im m D(q'^2)}{q'^2 - q^2}$$



$$D(q^2) = \sqrt{2}\, g_{NN\pi} \frac{1}{q^2 - m_\pi^2} f_\pi m_\pi^2 \, (+\cdots) \approx -2m_{\mathcal{N}}\, g_A(q^2) - q^2 g'_A(q^2)$$

Since there is no massless hadron, g'_A cannot develop a pole at $q^2 = 0$

$$\Rightarrow -D(0) = \boxed{2m_{\mathcal{N}} \ g_A(0) = \sqrt{2} \ g_{NN\pi}(0) \ f_\pi} \qquad (\pm 10\%)$$

Goldberger–Treimann relation

(ii) QFD:

$$\delta_{ab} f_\pi m_\pi^2 = \langle 0|D_5^a(0)|\pi^b(p)\rangle \underset{LSZ}{=} i \int d^4x \ e^{-ipx} (\partial^2 + m_\pi^2) \langle 0|T\{D_5^a(0)\phi_\pi^b(x)\}|0\rangle$$

$$= i \lim_{p^2 \to m_\pi^2} (m_\pi^2 - p^2) \int d^4x \ e^{-ipx} \langle 0|T\{D_5^a(0)\phi_\pi^b(x)\}|0\rangle$$

$$= i \lim_{p^2 \to m_\pi^2} \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} \int d^4x \ e^{-ipx} \langle 0|T\{D_5^a(0)\partial^\mu j_{5\mu}^b(x)\}|0\rangle$$

$$\partial^\mu T\{D_5^a(0) j_{5\mu}^b(x)\} = T\{D_5^a(0)\partial^\mu j_{5\mu}^b(x)\} + \delta(x^0) \left[j_{50}^b(x) D_5^a(0) - D_5^a(0) j_{50}^b(x) \right]$$

$$\Rightarrow \delta_{ab} f_\pi m_\pi^2 = i \lim_{p^2 \to m_\pi^2} \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} (ip^\mu) \int d^4x \ e^{-ipx} \langle 0|T\{D_5^a(0) j_{5\mu}^b(x)|0\rangle$$

$$+ i \lim_{p^2 \to m_\pi^2} \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} \int_{x^0 = 0} d^3\vec{x} \ e^{i\vec{p}\vec{x}} \langle 0|[D_5^a(0), j_{50}^b(x)]|0\rangle$$

PCAC: $\lim_{p^2 \to m_\pi^2} \approx \lim_{p_\mu \to 0}$ (up to 10%) $\left[(p^2 - m_\pi^2) \frac{F(m_\pi^2)}{p^2 - m_\pi^2} \approx F(0) \right]$

$$\Rightarrow \delta_{ab} f_\pi m_\pi^2 = -\frac{i}{f_\pi} \langle 0|[Q_5^b, D_5^a(0)]|0\rangle$$

with $\quad Q_5^b = \int_{x^0 = 0} d^3\vec{x} \ j_{50}^b(x)$

$D_5^a = \partial^\mu j_{5\mu}^a = (m_j + m_k)\bar{q}_j i\gamma_5 T_{jk}^a q_k = \bar{q} i\gamma_5 \{M, T^a\} q$

$Q_5^b = \int_{x^0=0} d^3\vec{x}\ q^\dagger \gamma_5 T^b q \qquad\qquad M_{ij} = m_i \delta_{ij}$

CR: $[Q_5^b, D_5^a(0)] = -i\bar{q}\{T^b, \{T^a, M\}\}q$

$\Rightarrow \delta_{ab} f_\pi^2 m_\pi^2 = -\langle 0|\bar{q}\{T^b, \{T^a, M\}\}q|0\rangle$

condensates: $\langle 0|\bar{q}_a q_b|0\rangle = \delta_{ab}\langle 0|\bar{q}_a q_a|0\rangle$

$\Rightarrow \delta_{ab} f_\pi^2 m_\pi^2 = -\langle 0|\bar{u}u|0\rangle\ Tr\{T^b, \{T^a, M\}\}$

$m_j \approx m_k \Rightarrow Tr\{T^b, \{T^a, M\}\} \approx \delta_{ab} \sum_k m_k$

$$\boxed{f_\pi^2 m_\pi^2 = -\sum_k m_k \langle \bar{u}u\rangle_0}$$ Gell–Mann/Oakes/Renner

f_π from decay $\pi^+ \to \mu^+ \nu_\mu$: $f_\pi = 94$ MeV

$\Rightarrow$ determination of quark masses and condensates
pion acquires mass through quark masses

(iii) generalization:

$$\langle z_1|\mathcal{O}(0)|z_2, \pi^a(p)\rangle \underset{PCAC}{\overset{LSZ}{=}} i\int d^4x\ e^{-ipx}(\partial^2 + m_\pi^2)\langle z_1|T\{\mathcal{O}(0)\phi_\pi^a(x)\}|z_2\rangle$$

$$= i \lim_{p^2\to m_\pi^2}(m_\pi^2 - p^2)\int d^4x\ e^{-ipx}\langle z_1|T\left\{\mathcal{O}(0)\frac{\partial^\mu j_{5\mu}^a(x)}{f_\pi m_\pi^2}\right\}|z_2\rangle$$

$$T\{A(0)\partial_\mu B^\mu(x)\} = \partial_\mu T\{A(0)B^\mu(x)\} + \delta(x^0)[A(0), B^0(x)]$$

$$\Rightarrow \langle z_1|\mathcal{O}(0)|z_2, \pi^a(p)\rangle = i \lim_{p^2\to m_\pi^2}\frac{m_\pi^2 - p^2}{f_\pi m_\pi^2}\left\{ip^\mu \int d^4x\ e^{-ipx}\langle z_1|T\{\mathcal{O}(0)j_{5\mu}^a(x)\}|z_2\rangle \right.$$
$$\left. + \int d^4x\ e^{i\vec{p}\vec{x}}\delta(x^0)\langle z_1|[\mathcal{O}(0), j_{50}^a(x)]|z_2\rangle\right\}$$

$$\lim_{p^2\to m_\pi^2} \approx \lim_{p_\mu\to 0} \Rightarrow \boxed{\begin{array}{c}\langle z_1|\mathcal{O}(0)|z_2, \pi^a(p)\rangle = \dfrac{i}{f_\pi}\langle z_1|[\mathcal{O}(0), Q_5^a]|z_2\rangle \\ \text{with } Q_5^a = \displaystyle\int_{x^0=0} d^3\vec{x}\ j_{50}^a(x)\end{array}}$$

$\Rightarrow$ soft-pion theorems

§4. Goldstone Theorem [Nuovo Cimento 19 (1961) 15]

$N =$ dimension of algebra belonging to the symmetry group of the full Lagrangian

$M =$ dimension of algebra that leaves vacuum invariant after symmetry breaking

$\Rightarrow$ there are $N - M$ massless goldstones in the theory

Proof:

$\mathcal{L}(\phi, \partial\phi)$ invariant under symmetry group

Noether currents conserved: $\partial_\mu V^\mu(x) = 0$

conserved charge: $Q = \int_{t=0} d^3\vec{x}\; V^0(0, \vec{r})$

$$\Rightarrow \boxed{[Q, \phi] = T\phi}$$

symmetry broken: $\langle 0|\phi|0\rangle = \langle\phi\rangle = v \neq 0$

$w_\mu(k) = \int d^4x\; e^{ikx} \langle 0|[V_\mu(x), \phi]|0\rangle$ fulfills:

symmetry condition: $k_\mu w^\mu(k) = 0$

symmetry broken:

$$\int dk^0\; w^0(k^0, \vec{0}) = \int dx^0 \int dk^0\; e^{ik^0x^0} \int d^3\vec{x}\; \langle 0|[V^0(x), \phi(0)]|0\rangle$$
$$= 2\pi\langle 0|[Q, \phi]|0\rangle = 2\pi T v \neq 0$$

$a = 1, \ldots, M : T^a v = 0$

$a = M + 1, \ldots, N : T^a v \neq 0 \quad \Rightarrow N - M$

spectral representation:

$$w_\mu(k) = (2\pi)^4 \sum_n \{\langle 0|V_\mu|n\rangle\langle n|\phi|0\rangle\ \delta_4(k-p_n)\ \Theta(k^0)$$
$$-\langle 0|\phi|n\rangle\langle n|V_\mu|0\rangle\ \delta_4(k+p_n)\ \Theta(-k^0)\}$$

→ Lorentz invariance: $\langle 0|V^\mu|n\rangle = f_n\, p_n^\mu$

→ positive energy spectrum: 1. sum: $\Theta(k^0) = \frac{1}{2}[1+\epsilon(k^0)]$

2. sum: $\Theta(-k^0) = \frac{1}{2}[1-\epsilon(k^0)]$

$$w_\mu(k) = k_\mu[\sigma_+(k^2) + \epsilon(k^0)\sigma_-(k^2)]$$

$$\sigma_\pm(k^2) = \frac{1}{2}(2\pi)^4 \sum_n \{\langle n|\phi|0\rangle\ f_n\,\delta_4(k-p_n) \pm \langle 0|\phi|n\rangle\ f_n^*\,\delta_4(k+p_n)\}$$

$$k_\mu w^\mu(k) = 0 \quad \Rightarrow \quad \boxed{k^2\sigma_\pm(k^2) = 0}$$

solution: $\sigma_\pm(k^2) = s_\pm\,\delta_1(k^2)$

$$w_\mu(k) = k_\mu[s_+ + s_-\epsilon(k^0)]\,\delta_1(k^2)$$

$$\int_{-\infty}^{\infty} dk^0\ w^0(k^0,\vec{0}) = s_- 2\int_0^\infty dk^0\ k^0\ \delta_1(k^2) = s_- \neq 0 \quad (N-M)$$

$$\Rightarrow \boxed{w_\mu(k) = [s_+ + s_-\epsilon(k^0)]\,\delta_1(k^2)\,k_\mu \qquad \text{with } s_- \neq 0}$$

→ there are $(N-M)$ states with $p_n^2 = 0 \Rightarrow \boxed{m=0}$

goldstones

q.e.d.